UNIT - II
MECHANICS

CENTRE OF GRAVITY

A rigid body may be considered to consist of a large number of small particles. Each particle is attracted towards the centre of the earth due to the gravitational force of the earth. These forces are proportional to the mass of the particles. These forces are called the weight of the particle. These forces are called the weight of the particle. The particles are very small when comparing to the earth. Hence the lines joining the centre of the earth with the particles are almost parallel forces. The resultant of these linear parallel forces is called the weight of the body. This is equa l to the sum of the weights of the component particles. This resultant will pass through a particular point of the body. This point is called the centre of gravity of the body.

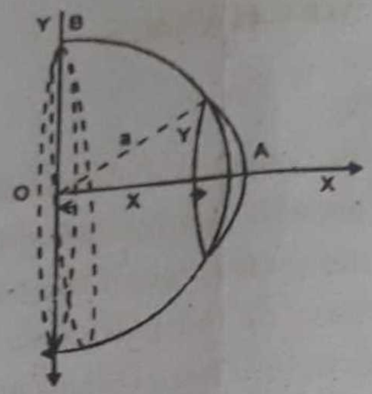
The centre of gravity of a body is defined as that point through which the resultant of the weight of all the particle of the body acts, whatever may be the position of the body.

Centre of gravity of a solid hemisphere

Let AB be the quadrant of the arc of a circle of radius 'a'. The centre O is taken as origin. If OA and OB are taken as axes, the equation of the curve AB is,

$$X^2 + Y^2 = a^2 \quad (1)$$

When the area AOB is rotated about the axis OA we will get the volume of the hemisphere. Due to the symmetry of the hemisphere, the centre of gravity lies on the axis OA. H e n c e $\bar{y} = 0$.



Consider a circular section of thickness dx at a distance x from O. Let the radius of the circular section be y.

Volume of the elementary section $= \pi y^2 dx$ - (2)

If ρ is the density, then the mass of this section $= \pi y^2 dx \rho$ - (3)

The centre of gravity of the circular disc will lie at (x, 0), its cetnre.

$$\bar{x} = \frac{\sum\limits_{x=0}^{x=a} \pi y^2 \rho dx.x}{\sum\limits_{x=0}^{x=a} \pi y^2 \rho dx}$$

$$= \frac{\int_0^a y^2 x\, dx}{\int_0^a y^2\, dx} \qquad (4)$$

But we know that $y^2 = (a^2 - x^2)$

$$\bar{x} = \frac{\int_0^a (a^2 - x^2)\, x\, dx}{\int_0^a (a^2 - x^2) dx}$$

$$= \frac{\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_0^a}{\left[a^2x - \frac{x^3}{3}\right]_0^a}$$

$$= \frac{\left[\frac{a^4}{2} - \frac{a^4}{4}\right]}{\left[a^3 - \frac{a^3}{3}\right]}$$

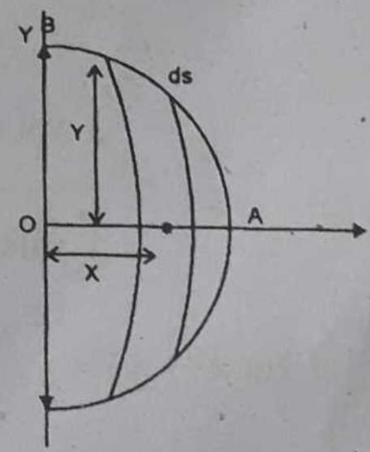
$$= \frac{a^4/4}{2a^3/3}$$

$$\bar{x} = 3a/8 \qquad - \qquad (5)$$

Therefore the centre of gravity of a solid hemisphere of radius 'a' lies on the axis at a distance 3a/8 from the centre.

Centre of gravity of hollow hemisphere

Let AB be the quadrant of the arc of a circle of radius 'a'. Let O be the centre and it is taken as origin.



If OA and OB are considered as axes, then the equation for the curve AB is.

$$x^2 + y^2 = a^2 \qquad - \qquad (1)$$

When the arc AB is rotated about the axis OA, it will generate the surface of the hemisphere. Due to symmetry the centre of gravity lies on OA. Therefore $\bar{y}$ = O.

Consider an elementary ring of thickness ds at a distance x from O. Let y be the radius of the ring.

Surface area of the elementary ring = $2\pi y ds$

If ρ is the mass per unit area (surface density), the mass of the elementary ring = $2\pi y \rho ds$

This mass is in the form of a circular ring. Hence, the centre of gravity is at (x, O), its centre.

$$\bar{x} = \frac{\sum_{x=0}^{x=a} 2\pi y \rho ds.x}{\sum_{x=0}^{x=a} 2\pi y \rho ds} \qquad (2)$$

$$= \frac{\int_0^a yx\, ds}{\int_0^a y ds} \qquad (3)$$

But $x^2 + y^2 = a^2$

Differentiating

$$2x + 2y.\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\left[\frac{ds}{dx}\right]^2 = 1 + \left[\frac{dy}{dx}\right]^2$$

$$= 1 + \frac{x^2}{y^2} = \frac{x^2 + y^2}{y^2}$$

$$\left[\frac{ds}{dx}\right]^2 = \frac{x^2}{y^2}$$

$$\frac{ds}{dx} = \frac{a}{y}$$

$$ds = \frac{x}{y} dx \qquad - \qquad (4)$$

Substituting equation (4) in equation (1), we get

$$\bar{x} = \frac{\int_0^a xy(a/y)dx}{\int_0^a y(a/y)dx}$$

$$\bar{x} = \frac{\int_0^a axdx}{\int_0^a adx}$$

$$= \frac{a(x^2/2)_0^a}{a(x)_0^a}$$

$$= \frac{a^3}{2} = \frac{1}{a^2} = \frac{a}{2}$$

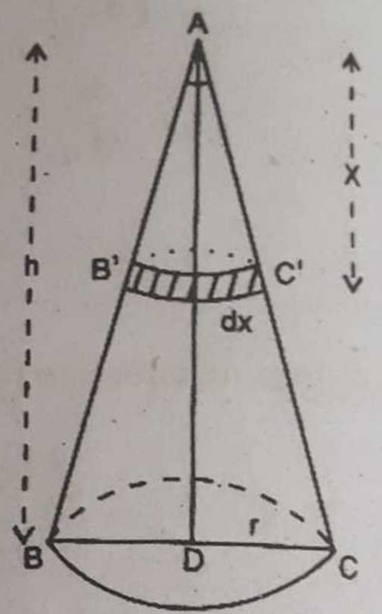
$$\bar{x} = a/2 \qquad - \qquad (5)$$

Thus the centre of gravity of a hollow hemisphere lies on its axis at a distance a/2 from the centre.

Centre of gravity of a solid cone

When the number of bases of a pyramid is increased indefinitely, the plane base becomes a circle. Hence the pyramid becomes a right solid cone of height 'h'. The centre of gravity of the cone lies at a depth of (3/4) h from the vertex.

The centre of gravity of a cone can be calculated as follows.



Let ABC represent the cross section of a right cone of height 'h'. Let 'r' be the radius of the base and D is the centre. AD = h → the vertical height of the cone. Consider a disc A'B' of thickness dx at a distance x from the vertex of the cone. This disc is parallel to the base. Let α be the semi – vertical angle of the cone.

If y is the radius of the elementary disc, then

$y = x \tan \alpha$ - (1)

Volume of the disc = Area x Thickness

$= \pi x^2 \tan^2 \alpha \, dx$ - (2)

If ρ is the density of cone, the mass of the disc

$= \pi x^2 \tan^2 \alpha \, dx \, \rho$ - (3)

The moment of the disc $= \pi x^2 \rho \tan^2 \alpha \, dx.x$ - (4)

In a similar way we can divide the cone into different disc and the moments of the discs can be calculated. The sum of the moments of all the discs about the point A is,

$$= \int_0^h \pi \rho \tan^2\alpha \, x^3 \, dx \qquad \text{- (5)}$$

$$= \frac{\pi \rho \tan^2\alpha \, h^4}{4} \qquad \text{- (6)}$$

Volume of the cone $= \int_0^h \pi \tan^2\alpha \, x^2 \, dx$

$$= \frac{\pi \tan^2\alpha \, h^3}{3} \qquad \text{- (7)}$$

Mass of the cone $= \dfrac{\pi \tan^2\alpha \, h^3 \rho}{3}$ - (8)

Let us consider the centre of gravity of the cone is at a distance $\overline{x}$ from the vertex.

Moment of the mass of the cone $= \dfrac{\pi \rho \tan^2\alpha \, h^3 \overline{x}}{3}$ - (9)

$$\frac{\pi \rho \tan^2\alpha \, h^3 \overline{x}}{3} = \frac{\pi \rho \tan^2\alpha \, h^4}{4}$$

$$\overline{x} = \frac{3}{4} h \qquad \text{- (10)}$$

Therefore the centre of gravity of the cone is on its axis at a distance (3/4)h from the vertex.

STATES OF EQUILIBRIUM

If a marble M is placed on a curved surface of a bowl S, it rolls down and settles in equilibrium at the lowest point A (Fig. a). This equilibrium position corresponds to minimum potential energy. If the marble is disturbed and displaced to a point B, its energy increases When it is released, the marble rolls back to A. Thus the marble at the position A is said to be in stable equilibrium.

being displaced, the bar will remain in its new position, on being released, and the equilibrium is said to be neutral.

FLOATATION

Consider a body floating in a liquid. When it is in equilibrium two vertical forces act on the floating body. They are,

1. The weight of the body acts vertically downward through the centre of gravity.
2. The resultant vertical thrust acts on the floating body. It is equal to the weight of the floating body and its acts through the centre of buoyancy. (The centre of buoyancy is the centre of gravity of the displaced liquid).

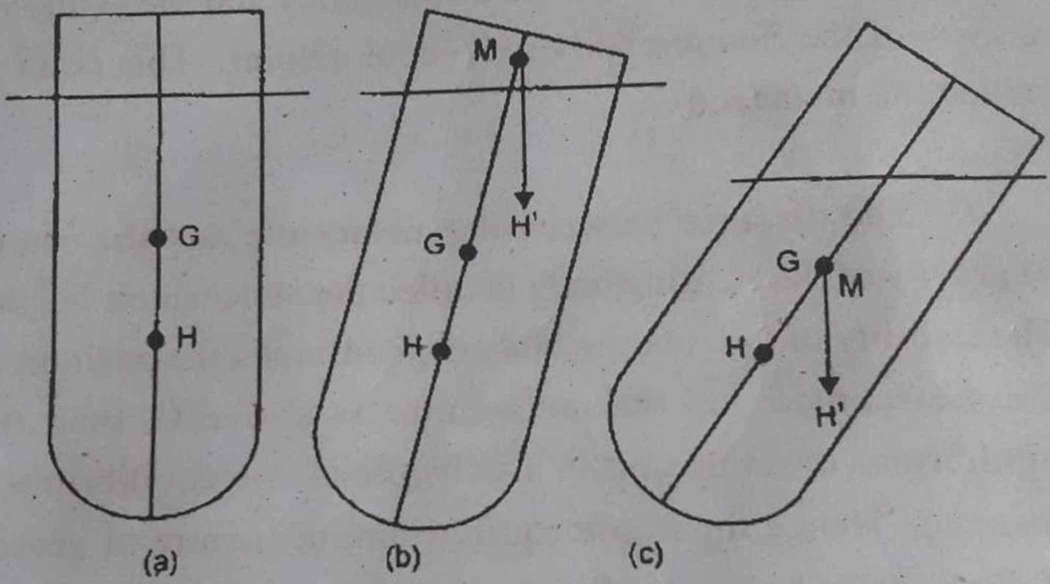
For the equilibrium of the floating body, the two forces should be equal and should act in a vertical line in opposite direction. Hence the laws of floatation can be stated as follows.

1. When a body floats freely in a liquid, the weight of the displaced liquid is equal to the weight of the floating body.
2. The centre of gravity of the floating body and the centre of gravity of the displaced liquid act in the same vertical line.

a. Stability of floating bodies

When a body floats in a liquid, the C.G. of the body G and the centre of buoyancy should lie in the same vertical line.

When a floating body is slightly displaced, the line HG will be inclined to the vertical. If the thrust of the newly displaced liquid tries to return the body to the original equilibrium position, then it is called stable, equilibrium. On the other hand, if the thrust turns the body further, then it is called unstable equilibrium. These are shown in figure. Figure (a) shows the initial state. Figure (b) and figure (c) show the displaced position. In these 'H' is the new centre of buoyancy. The vertical line HM meets the line HG at M. M is called the metacentre of the floating body and the height GM is called metacentric height.



If figure (b) the metacentre M lines above G. Now the force acts in a direction opposite to that of the turning. So, the body returns to its original equilibrium position. This type of equilibrium is called stable equilibrium.

In figure (c) the metacentre M lies below G. Hence the force on the body turns the body further. So the body will not return to its equilibrium position. This type of equilibrium is called unstable equilibrium.

Hence for a stable equilibrium of a floating body, the metacentre must be always above the centre of gravity of the body.

b. Metacentre

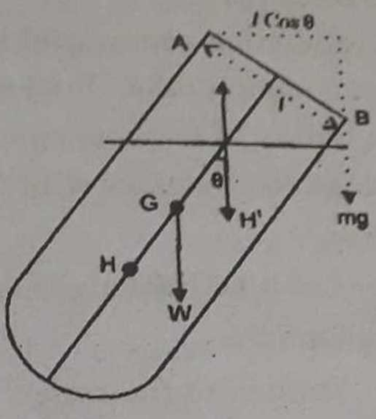
Consider a freely floating body in a liquid. Let it be slightly distributed from its position. Let us consider that there is no change in the volume of the displaced liquid. Now the vertical line through the new centre of buoyancy and the line passing through the old centre of buoyancy and the centre of gravity of the floating body will cut at a point. This point is called the metacentre.

The distance between the metacentre and the centre of gravity of the floating body is called the metacentric height. The stability of the floating body depends upon the position of the metacentre. If the metacentre is above G, then the equilibrium is stable and if it is below G, the equilibrium is unstable. Hence for stable equilibrium, the centre of gravity of the floating body should be as below as possible.

c. Determination of metacentric height of a ship

The weight of the ship is determined by the displacement method. At the two end of the ship A and B, two identical boats are attached. The weight of the water is calculated from the volume of the water filled in the boat. Let *l* be the distance between A and B. Filling the boats A and B alternatively with water is equivalent to moving a weight mg from A to B across the deck. Filling the boat B with the same mass of water as in A. Using the plumb line suspended in the ship, the tilting angle θ may be measured.

Due to the tilting of the ship, there is a small change in the metacentre. Let H and H' be the old and new metacentre. G the centre of gravity of the ship and GM the metacentric height.



The deflecting couple due to alternatively filling of water

$= mgl \cos\theta$ - (1)

The weight of ship 'W' acts downward through G and an upward thrust acts at the new centre of buoyancy. These two forces constitutes a couple.

The restoring couple $= WGM \sin\theta$ - (2)

For the equilibrium,

Restoring couple = Deflecting couple

$$WGM \sin\theta = mgl \cos\theta \qquad - (3)$$

$$GM = \frac{mgl}{W. \tan\theta} \qquad - (4)$$

Since the value of θ is small, $\tan\theta = \theta$

$$GM = \frac{mgl}{W\,\theta} \qquad - (5)$$

Using equitation (5) the metacentric height of the ship can be calculated.