Complex Variables

6·1. Complex Numbers

A number of the form $a + ib$ where $i = \sqrt{(-1)}$ and a and b are real numbers is called a complex number. Complex numbers first became necessary in the sutdy of algebraic equations. It is well known that every quadratic equation has two roots, every cubical equation has three roots and so on. If only real numbers are considered, then the equation $x^2 + 1 = 0$ has no root and $x^3 - 1 = 0$ has only one root ; while the former being a quadratic equation must have two roots and the latter being a cubical equation must have three roots. Thus solely the real numbers are not sufficient for all mathematical needs. Euler was the first mathematician who introduced the symbol i for $\sqrt{(-1)}$ with the property $i^2 = -1$ and accordingly the roots of the equations $x^2 + 1 = 0$ and $x^3 - 1 = 0$ in terms of symbol i were given. Later on Hamilton, Gauss, Cauchy, Riemann and Wierstrass and other great mathematicians extended the field of real numbers to the still larger field of complex numbers. Although the complex numbers are capable of a geometrical interpretation, it becomes necessary to give their definition in terms of real numbers.

By choosing one of the several possible lines of procedure *we define a complex number as an ordered pair of real numbers* like (x, y). If we write complex number $z = (x, y) = x + iy$; where x, y are real numbers, then x is called the *real* and y the *imaginary part* of the complex number z, Usually the real part x is denoted by the R_z or $R(z)$ and the imaginary part y by I_z or $I(z)$. A pair of the type $(0, y)$ is a purely imaginary number.

Two complex numbers are said to be equal if and only if their real parts are equal and their imaginary parts are equal *e.g.*

$$(x_1, y_1) = (x_2, y_2) \text{ if and only if } x_1 = x_2 \text{ and } y_1 \text{ and } y_2.$$

6·2. Review of Algebraic Operations of Complex Numbers

1. Addition. The sum of two complex numbers

$z_1 = x_1 + y_1 = (x_1, y_1)$ and $z_2 = x_2 + iy_2 = (x_2, y_2)$ is defined by the equality

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

$$= (x_1 + x_2, y_1 + y_2) \quad \text{... (1)}$$

i.e., the sum of two complex numbers is a complex number with its real part as the sum of real parts and its imaginary part as the sum of imaginary parts of given numbers.

2. Subtraction. The difference of two complex numbers

$z_1 = x_1 + iy_1 = (x_1, y_1)$ and $z_2 = (x_2 + iy_2) = (x_2 + y_2)$ is defined by the equality

$$z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + i(y_1 - y_2)$$

$$= (x_1 - x_2, y_1 - y_2) \quad \text{... (2)}$$

i.e., the difference of two complex numbers is a complex number with its real part as the difference of real parts and its imginary part as the difference of imaginary parts of given complex numbers.

3. Multiplication. The product of two complex numbers

$z_1 = x_1 + iy_1 = (x_1, y_1)$ and $z_2 = x_2 + iy_2 = (x_2, y_2)$ is defined by the equality

$$z_1 . z_2 = (x_1 + iy_1) . (x_2 + iy_2) = (x_1x_2 - y_1y_2) + i\,(x_1y_2 - x_2y_1)$$
$$= (x_1x_2 - y_1y_2, x_1y_2 - x_2y_1) \quad \text{... (3)}$$

4. Division. If $z_1 = (x_1 + iy_1) = (x_1, y_1)$ and $z_2 = (x_2 + iy_2) = (x_2, y_2)$ are two complex numbers, then division of z_1 by z_2 is defined as

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = (x_1 + iy_1)\,(x_2 + iy)^{-1} = (x_1, y_1)\,(x_2, y_2)^{-1}$$

provided that $(x_2, y_2) \neq (0, 0)$

where $(x_2, y_2)^{-1}$ is multiplicative inverse of (x_2, y_2) *i.e.*

$$(x_2, y_2) = \left(\frac{x_2}{x^2{}_2 + y^2{}_2}, \frac{-y_2}{x^2{}_2 + y^2{}_2}\right)$$

6·3. Complex Conjugates

The complex conjugate or simply the conjugate of a complex number $z = (x, y) = x + iy$ is defined as a complex number $x - iy$ and is usually denoted by $\bar{z}$ or z^* *i.e.*

$$\bar{z} = x - iy$$

If $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$, then $z_1 + z_2 = (x_1 + x_2, y_1 + y_2)$

therefore $\overline{z_1 + z_2} = (x_1 + x_2) - i\,(y_1 + y_2) = (x_1 - iy_1) + (x_2 - iy_2)$

i.e., $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

That is, *the complex conjugates of the sum of given complex numbers is the sum of their complex conjugates.*

In the like manner is may easily be verified that the operation of taking conjugates is also distributive with respect to subtraction, multiplication and division, *i.e.*,

$$\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$$
$$\overline{z_1 z_2} = \bar{z}_1\,\bar{z}_1$$
$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}, \; z_2 \neq 0$$

The conjugate of a real number is the number itself. It may also be noted that

$$z + \bar{z} = (x + iy) + (x - iy) = 2x = 2R\,(z)$$

i.e., the sum of a complex number and its conjuate is real number

Further, $z - \bar{z} = (x + iy) - (x - iy) = 2iy = 2(I_z)\,i$

i.e., the difference of a complex number and its conjugate is purely imaginary number.

6·4. Modulus and Argument of a Complex Number

Let $z = (x, y)$ be a complex number. The *modulus* or the *absolute value* of z is denoted by $|z|$ and defined as $\sqrt{x^2 + y^2}$.

Let r be any non-negative number and θ any real number. If we take

$$x = r\cos\theta,\; y = r\sin\theta, \text{ then}$$
$$r = +\sqrt{x^2 + y^2} \text{ and } \theta = \tan^{-1}\frac{y}{x}.$$

The non-negative quantity $r = +\sqrt{x^2 + y^2}$ is the modulus of z. The real quantity θ is called the *argument* or the *amplitude* of z and is denoted by arg z or amp z.

From $|z| = \sqrt{x^2 + y^2}$ it follows that $|z| = 0$ if and only if $x = 0$ and $y = 0$. Also, we have

$$z = x + iy = r(\cos\theta + i\sin\theta) = r[\cos(2n\pi + \theta) + i\sin(2n\pi + \theta)$$

$$n = 0, \pm 1, \pm 2, \ldots$$

From $\arg z = \theta \tan^{-1}\frac{y}{x}$, it follows that the argument of a complex number is not unique since if θ is the value of the argument, then $2n\pi + \theta$ $(n = 0, \pm 1, \pm 2, \ldots)$ are also the values of the agrument. Thus, argument θ of a complex number z can have infinite number of values which differ from each other by 2π or by any multiple of 2π. The value θ of arg z such that $-\pi \leq \theta \leq \pi$ is called the *principal value* of the argument.

Properties of Moduli. The moduli of the complex numbers possess the following properties which may be easily verified :

1. *The modulus of the sum of complex numbers can never exceed the sum of their moduli, i.e.,* if z_1 and z_2 are two complex numbers, then

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

2. *The modulus of the difference of two complex numbers can never be less than the difference of their moduli, i.e.,*

$$|z_1 - z_2| \geq |z_1| - |z_2|$$

3. *The modulus of the product of two complex numbers is the product of their moduli, i.e.,*

$$|z_1 z_2| = |z_1||z_2|$$

4. *The modulus of the quotient (division) of two complex number is the quotient (division) of their moduli, i.e.,*

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}.$$

Properties of Arguments. The arguments of the complex numbers possess the following properties which may be easily verified.

1. *The argument of the product of any number of complex numbers is equal to the sum of their arguments i.e.,* if $z_1, z_2, \ldots, z_n$ are complex numbers, then

$$\arg(z_1 z_2 \ldots z_n) = \arg z_1 + \arg z_2 + \ldots + \arg z_n$$

2. *The arguments of the quotient (division) of two complex numbers is equal to the difference of their arguments, i.e.,*

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$$

6·5. Graphical Representation on Argand Diagram and Trigonometric Form

Although complex numbers are essentially algebraic quantities, they may be given a convenient geometrical interpretation. Consider a complex number $z = (x, y)$. This form suggests that the complex number $z = (x, y)$ may be represented by a point P in the x-y plane whose coordinates are (x, y) referred to rectangular cartesian axes OX and OY. The origin represents the point $z = 0$. When used for the purpose of representing complex numbers z geometrically the x-axis is called the *axis of reals* or the *real axis*, the y-axis is called the *axis of imaginaries* or *imaginary axis* and the x-y plane is called Argand plane or Gaussian plane or complex plane or the

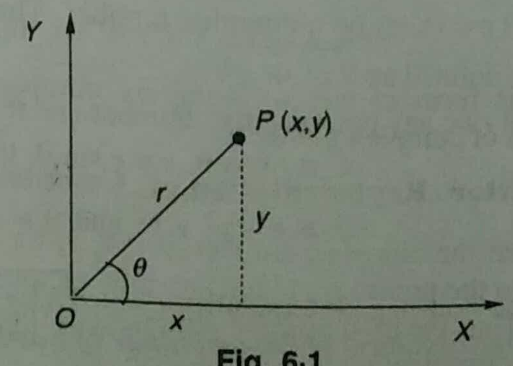

Fig. 6·1

z-plane. To each complex number there corresponds one and only one point in the complex plane and conversely to each point in the complex plane there corresponds one and only one complex number. Because of this the complex number z in the complex plane is often referred to as the point z. If we introduce polar coordinates, we have

$$x = r\cos\theta, \qquad y = r\sin\theta$$

Then $$z = x + iy = r(\cos\theta + i\sin\theta) \qquad \text{... (1)}$$

This represents the *polar form* of the complex number z. The number $r = \sqrt{x^2 + y^2}$, which is always positive, is the *modulus* of the complex number and is equal to the length of the line OP. The angle $\theta \left(= \tan^{-1}\dfrac{y}{x}\right)$ is the *argument* of complex number z.

It is well known that $\cos\theta$, $\sin\theta$ and e^t have the following Maclaurin expansions :

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + - \ldots$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + - \ldots$$

$$e^t = 1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots$$

Hence we have

$$\cos\theta + i\sin\theta = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots\right)$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} + \ldots$$

$$= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \ldots$$

where $i^2 = -1$, $i^3 = -i$, $i^4 = 1$ etc.

So that $$\cos\theta + i\sin\theta = e^{i\theta} \qquad \text{... (2)}$$

Similarly $$\cos\theta - i\sin\theta = e^{-i\theta} \qquad \text{... (3)}$$

From (2) and (3) it follows immediately that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \text{ and } \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \qquad \text{... (4)}$$

These equations represent the *relation between exponential and trigonometric functions.*

Using equations (1) and (2) a complex number z may be written in the convenient form as

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta} \qquad \text{... (5)}$$

This form of writing complex numbers is very convenient in the multiplication and division of complex numbers.

Vector Representation of Complex Numbers. Sometimes it is found useful to represent the complex number $z = (x, y)$ by the directed line segment or vector $\overrightarrow{OP}$ from the origin to the point (x, y) ; also by any vector obtained by translating that vector in the complex plane. Thus the modulus and argument of the complex number give the magnitude and direction of the corresponding vector and vice-versa.

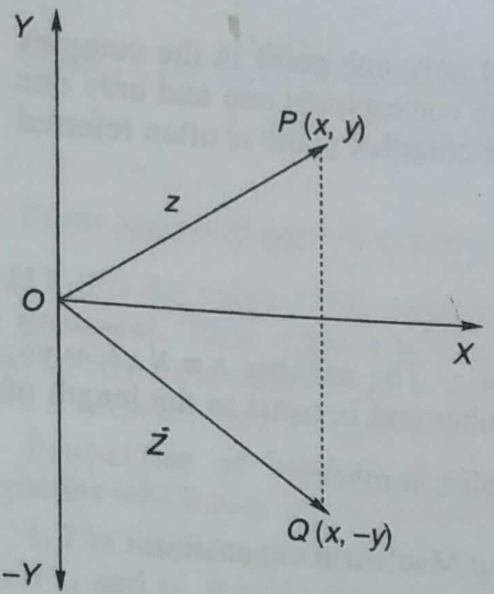

Fig. 6·2

Both the point representation and vector representation of complex numbers are very useful.

Representation of Complex Conjugate. The complex conjugate of complex number $z = x + iy = (x, y)$ is $(x - iy) = (x, -y)$. Geometrically, if complex number z is represented by point P, then its conjugate $\bar{z}$ is represented by the reflection or image Q of P in the real (x) axis. If (r, θ) are the polar coordinates of complex point P, then the polar coordinates of conjugate point Q are $(r, -\theta)$ so that

$$\arg z = -\arg \bar{z}.$$

Ex. 1. *Graph the region represented by $1 < |z + i| \leq 2$ on the Argand diagram.*

(Delhi 2005)

Solution. The region is enclosed between two circles of radii between $R = 1$ and $R = 2$ with centre at $z = -i = (0 - 1.i)$ or with centre on imaginary axis at $y = -1$; Accordingly the graph is shown in fig.

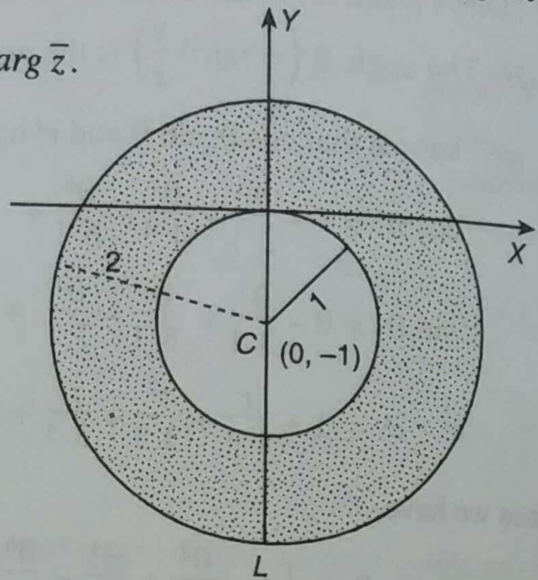

Fig. 6·3

Ex. 2. *Separate $\log_e z$ into real and imginary parts.*

(Delhi 1999)

Solution. We have $z = x + iy$

Let $x = r \cos \theta$ ...(1)

$y = r \sin \theta$ ... (2)

Squaring and adding (1) and (2), we have

$$x^2 + y^2 = r^2 \Rightarrow r = \sqrt{x^2 + y^2} \quad \text{... (3)}$$

Dividing (2) by (1), we get

$$\tan \theta = \frac{y}{x} \Rightarrow \theta = \tan^{-1} \frac{y}{x} \quad \text{... (4)}$$

$$\therefore \quad \log z = \log_e (x + iy) = \log_e \{r (\cos \theta + i \sin \theta)\}$$

$$= \log_e r + \log_e \{\cos (2n\pi + \theta) + i \sin (2n\pi + \theta)\}$$

$$= \log_e r + \log_e e^{i(2n\pi + \theta)}$$

$$= \log_e r + i (2n\pi + \theta)$$

$$= \log_e \sqrt{(x^2 + y^2)} + i\left(2n\pi + \tan^{-1}\frac{y}{x}\right)$$

For $n = 0$

$$\therefore \quad \log_e z = \log_e \sqrt{(x^2 + y^2)} + i\left[2n\pi + \tan^{-1}\left(\frac{y}{x}\right)\right].$$

This is required expression.

Ex. 3. *Write $i^{(1-i)}$ in the form $r (\cos \theta + i \sin \theta)$*

Solution $\quad i^{1-i} = e^{\log i^{(1-i)}} = e^{(1-i) \log i}$

$$= e^{(1-i) \log (\cos \frac{\pi}{2} + i \sin \frac{\pi}{2})}$$

$$= e^{(1-i)\log\{\cos(2n\pi+\frac{\pi}{2})+i\sin(2n\pi+\frac{\pi}{2})\}}$$

$$= e^{(1-i)\log e^{i(2n\pi+\frac{\pi}{2})}}$$

$$= e^{(1-i)\,i\,(2n\pi+\frac{\pi}{2})} = e^{(i+1)\,(2n\pi+\frac{\pi}{2})}$$

$$= e^{i(2n\pi+\frac{\pi}{2})} \,.\, e^{(2n\pi+\frac{\pi}{2})}$$

$$= e^{(2n\pi+\frac{\pi}{2})}\{\cos(2n\pi+\frac{\pi}{2})+i\sin(2n\pi+\frac{\pi}{2})\}$$

$$= e^{(2n\pi+\frac{\pi}{2})} \,.\, i = i\, e^{(2n\pi+\frac{\pi}{2})}$$

Ex. 4. *Obtain the principal value of log $(\sqrt{3}-i)$* (Delhi 1999)

Solution. $$(\sqrt{3}-i) = 2\left(\frac{\sqrt{3}}{2}-i\frac{1}{2}\right) = 2\{\cos(2n+\tfrac{1}{6})\pi - i\sin(2n+\tfrac{1}{6})\pi\}$$

$$= 2e^{-i(2n+\frac{1}{6})\pi}$$

$$\therefore \quad \log_e(\sqrt{3}-i) = \log_e\left\{2e^{-i(2n+\frac{1}{6})\pi}\right\} = \log_e 2 + \log e^{-i(2n+\frac{1}{6})\pi}$$

$$= \log_e 2 - i\,(2n+\tfrac{1}{6})\pi$$

This is general value of $\log_e(\sqrt{3}-i)$.

The principal value is obtained by putting $n = 0$;

therefore principal value of $\log_e(\sqrt{3}-i)$

is $\left\{\log_e 2 - i\dfrac{\pi}{6}\right\}$

Ex. 5. *Find the general value of log (1 + i) + log (1 – i)*

Solution. $$1+i = \sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)$$

$$= \sqrt{2}\left\{\cos\left(2n+\frac{1}{4}\right)\pi + i\sin\left(2n+\frac{1}{4}\right)\pi\right\}$$

$$= \sqrt{2}\,e^{(2n+\frac{1}{4})\pi}$$

$$\log(1+i) = \log\{\sqrt{2}\,e^{(2n+\frac{1}{4})\pi}\} = \log\sqrt{2} + i\,(2n+\tfrac{1}{4})\pi$$

$$\text{Similarly } (1-i) = \sqrt{2}\,e^{+i(2n-\frac{1}{4})\pi}$$

$$\log(1-i) = \log\left\{\sqrt{2}\,e^{+i(2n-\frac{1}{4})\pi}\right\} = \log\sqrt{2} + i\,(2n-\tfrac{1}{4})\pi$$

$$\therefore \quad \log(1+i)+\log(1-i) = \log\sqrt{2} + i\,(2n+\tfrac{1}{4})\pi + \log\sqrt{2} + i\,(2n-\tfrac{1}{4})\pi$$

$$= 2\log\sqrt{2} + 4n\pi i = \log 2 + 4n\pi i$$

6·6. Some Definitions Underlying Complex Analysis

Set of Points. Any collection of points in a complex plane is called an aggregate or a set of points.

Neighbourhood of a Point. A neighbourhood of point z_0 in the Argand plane means the set of all point z such that $|z - z_0| < \epsilon$, where ϵ is any arbitrarily chosen small positive number.

Limit Point. A point z_0 is said to be the limit point of a set of points S in the Argand plane if every neighbourhood of z_0 contains points of set S other than z_0. Thus, each point on the circumference of the circle $|z| = r$ is a limit point of the set $|z| < r$ and these limit points do not belong to the set. Thus *limit points of a set are not necessarily the points of the set.* Accordingly, there are two types of limit points.

(*i*) **Interior Points.** A limit point z_0 of the set S is an interior point if in the neighbourhood of z_0 there exist entirely the points or the set S.

(*ii*) **Boundary Points.** A limit point z_0 which is not an interior point is said to be the boundary point.

Closed set. If all the limit points of a set belong to the set, then the set is said to be closed.

Open set. A set which consists entirely of interior points is said to be an open set.

Bounded and Unbounded sets. A set of points is said to be *bounded* if there exists a positive number k such that $|z| \le k$ is satisfied for all points z of the set. If there does not exist such number k, then the set is said to be *unbounded*.

Compact set. A set is called *compact* if it is bounded as well as closed.

Convex or Connected set. A set of points is said to be convex or connected if every pair of its points can be joined by some continuous chain of a finite number of line segments (*i.e.* by a polygonal arc) which consists only of the points of the set. Thus the set consisting of all points interior to the circle $|z| = 1$ and all points exterior to circle $|z| = 2$ is not connected.

Domain. A connected open set is called an *open region* or *domain*.

Closed domain. The set, obtained by adding to an open connected set its boundary points is called a *closed domain* or *region*.

6·7. Functions of a Complex Variable

A complex variable is an ordered pair of real variables, *i.e.*,

$$z = (x, y) = x + iy$$

Let $z = x + iy$ and $w = u + iv$ be two complex variables. If to every value of z in certain domain D, there corresponds one or more values of w in a well-defined way, then w is said to be the function of complex variables z on the domain D and is written as

$$w = f(z)$$

As u and v are both functions of x and y, this definition implies that a function of a complex variable z is exactly the same thing as complex function

$$u(x, y) + iv(x, y) \text{ of two real variables } x \text{ and } y.$$

If to each value of z, there corresponds only one value of w, then the function $w(z)$ is called the *single-valued function* of z. If the function $w(z)$ takes more than one value corresponding to a value of z, then $w(z)$ is said to be a *multi-valued (or many-valued) function* of z. Let us assume that *the term function signifies a single-valued function unless the contrary is clearly indicated*. Most of our work with multi-valued functions, such as $z^{1/2}$ can be carried out conveniently by dealing with single-valued functions, each of which takes on just out of the multiple values for each value of z in a specified domain.

6·8. Limit, Continuity and Differentiability

Limit. Let $\omega = f(z)$ be any function of complex variable z defined in the neighbourhood of z_0 except perhaps at z_0 itself. Then the function $f(z)$ is said to have limit l as z approaches z_0 if for every positive real number ϵ (however small but not zero) there exists a positive real number δ (usually depending on ϵ) such that

$|f(z) - l| < \epsilon$ for all z, z_0 ; $z \neq z_0$ in the entire domain $0 < |z - z_0| < \delta$

This means that the values of $f(z)$ are as close as desired to l for all z which are sufficiently close to z_0 (Fig. 6·4). The statement that the limit of the function $f(z)$, as z approaches z_0 is a number l and is expressed as

$$\lim_{z \to z_0} f(z) = l$$

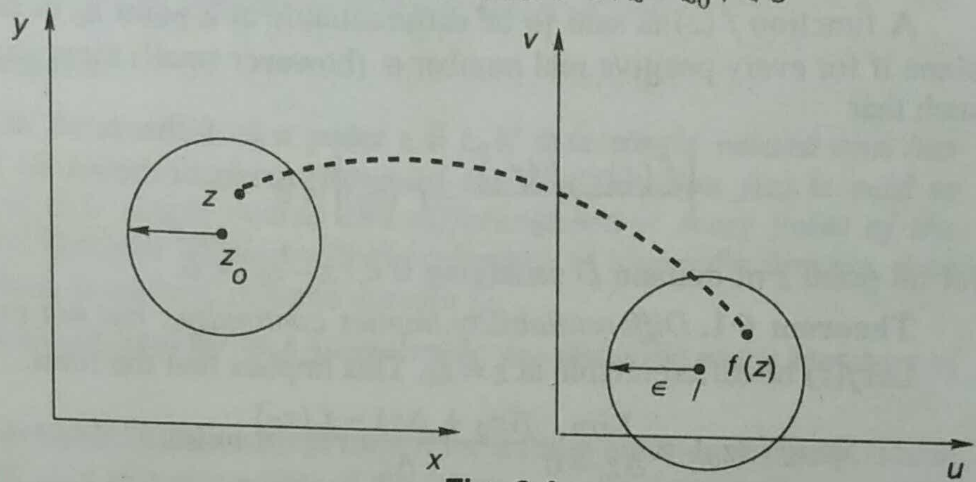

Fig. 6·4

This definition of the limit implies that z may approach z_0 in an arbitrary manner, not from some particular direction in the complex plane.

Continuity. Let $f(z)$ be any function of complex variable z defined in the closed domain D. Then the function $f(z)$ is said to be continuous at the point z_0 of domain D if for every positive real number ϵ (however small but not zero) there exists a positive real number $\boldsymbol{\delta}$ such that

$$|f(z) - f(z_0)| < \epsilon$$

for all points z of the domain D satisfying $0 < |z - z_0| < \delta$. From this definition it follows that the function $f(z)$ is *continuous* at $z = z_0$ *iff* $f(z_0)$ is defined and

$$\lim_{z \to z_0} f(z) = f(z_0)$$

A *continuous function,* without further qualification, is one which is continuous at all points where it is defined.

The real number δ generally depends on ϵ and z_0. If δ is independent of z_0 or in other words if there exists a number $h(\epsilon)$, independent of z_0, such that $|f(z) - f(z_0)| < \epsilon$ holds for every pair of points z, z_0 of the domain D for which $|z - z_0| < h$, then $f(z)$ is said to be uniformly continuous in domain D.

Differentiability. Let $f(z)$ be any single-valued function of complex variable z defined in the domain D of the Argand plane. Then the function $f(z)$ is said to be differentiable at a point z_0 of domain D if the limit

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

exists. This limit is then called the *derivative* of $f(z)$ at point $z = z_0$.

Setting $z_0 + \Delta z = z$, we have $\Delta z = z - z_0$, we may define the differentiability as follows :

The function $f(z)$ is said to be *differentiable* at a point z_0 of domain D if the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists provided that z is also a point of domain D. This limit is then called the derivative of $f(z)$ at a point $z = z_0$.

This definition implies that z may approach z_0 from any direction. Hence the differentiability means that along whatever path z approaches z_0, the derivative $f'(z_0)$ always aproaches a *unique value*.

In a more elementary form the differentiability may be redefined as follows :

A function $f(z)$ is said to be differentiable at a point z_0 in the domain D of the Argand plane if for every positive real number $\in$ (however small) there exists a positive real number δ such that

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \in$$

for all point z of domain D satisfying $0 < |z - z_0| < \delta$.

Theorem 6·1. *Differentiability implies continuity ; but not conversely.*

Let $f(z)$ be differentiable at $z = z_0$. This implies that the limit.

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \text{ exists}$$

$$\therefore \quad \lim_{\Delta z \to 0} \left[f(z_0 + \Delta z) - f(z_0) \right] = \lim_{\Delta z \to 0} \left[\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \cdot \Delta z \right]$$

$$= \lim_{\Delta z \to 0} \left[\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \right] \cdot \lim_{\Delta z \to 0} \Delta z = f'(z_0) \cdot 0 = 0$$

i.e.,

$$\lim_{\Delta z \to 0} f(z_0 + \Delta z) = f(z_0)$$

which shows that *the function $f(z)$ is continuous* at $z = z_0$

To show that converse may not be true, we consider the function

$$f(z) = |z|^2 = x^2 + y^2$$

The function is continuous everywhere due to continuity of $x^2 + y^2$.

But $$f'(z_0) = \lim_{\Delta z \to 0} \frac{|z_0 + \Delta z|^2 - |z_0|^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)(\bar{z}_0 + \Delta \bar{z}) - z_0 \bar{z}_0}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\bar{z}_0 \Delta z + z_0 \Delta \bar{z} + \Delta z \Delta \bar{z}}{\Delta z} = \lim_{\Delta z \to 0} \left(\bar{z}_0 + z_0 \cdot \frac{\Delta \bar{z}}{\Delta z} + \Delta \bar{z} \right)$$

$$= \lim_{\Delta z \to 0} \left(\bar{z}_0 + z_0 \cdot \frac{\Delta \bar{z}}{\Delta z} \right) \text{ since } \Delta \bar{z} \to 0 \text{ as } \Delta z \to 0$$

If $z_0 = 0$, then the derivative $f'(0) = 0$)

But if $z_0 \neq 0$, then let $\Delta z = \Delta r (\cos\phi + i \sin\phi)$; so that

$$\frac{\Delta \bar{z}}{\Delta z} = \frac{\Delta r (\cos\phi - i \sin\phi)}{\Delta r (\cos\phi + i \sin\phi)} = \frac{e^{-i\phi}}{e^{i\phi}} = e^{-2i\phi} = \cos 2\phi - i \sin 2\phi$$

Here $\phi = \arg(\Delta z)$

Now $\lim_{\Delta z \to 0} \frac{\Delta \bar{z}}{\Delta z}$ does not tend to a unique limit since it depends on $\arg(\Delta z)$. In other words, $\lim_{\Delta z \to 0} \frac{\Delta \bar{z}}{\Delta z}$ does not exist. This implies that $f'(z_0) = \lim_{\Delta z \to 0} \left(\bar{z}_0 + z_0 \cdot \frac{\Delta \bar{z}}{\Delta z} \right)$ does not exist. Hence *the function* $f(z) = |z|^2$, though continuous everywhere, *is differntiable anywhere* except at $z_0 = 0$. Hence the theorem.

Note. *All the familiar rules of real differential calculus, such as the rules for differentating a constant, integal powers of z, sums, differences, products and quotients of differential functions and the chain rule for differentiating a function, continue to hold in complex analysis.*

6·9. Definition : Analytic function

A function f (z) is said to be analytic at a point $z = z_0$ if it is single valued and has a derivative at every point in some neighbourhood of z_0. The function f(z) is said to be analytic in a domain D if it is single valued and differentiable at every point of the domain D. If the term *analytic function* is used without reference to a specific domain, then this term means a function which is analytic in some domain *D*.

Instead of term *analytic* the term *regular* and *holomorphic* are also used in the literature of the complex analysis.

A function may be differentiable in a domain *D* except for a finite number of points. These points are called the *singularities* or *singular points* of the function in the domain *D*.

6·10. The Necessary and Sufficient Conditions for f(z) to be Analytic : Cauchy-Riemann Differential Equations

The necessary conditions

Let $w = f(z) = u(x, y) + iv(x, y)$ $\qquad [\because z\ (x, y) = x + iy]$

From definition, the function $f(z)$ is analytic at point z if the limit $f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$ exists and always approaches a unique value as Δz (or Δx and Δy) $\to 0$ in any manner.

As $z = x + iy$, therefore, $\Delta z = \Delta x + i\,\Delta y$; so that we may write

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta x \to 0, \Delta y \to 0} \left[\frac{u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y)\} - \{u(x, y) + iv(x, y)}{\Delta x + i\,\Delta y} \right]$$

If we take Δz to be wholly real so that $\Delta y = 0$; then we get

$$f'(z) = \lim_{\Delta x \to 0} \left[\frac{\{u(x + \Delta x, y) + iv(x + \Delta x, y)\} - \{u(x, y) + iv(x, y)\}}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} \right] + i \lim_{\Delta x \to 0} \left[\frac{v(x + \Delta x, y) - v(x, y)}{\Delta x} \right]$$

$$= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \qquad \text{...(1)}$$

Since $f'(z)$ exists, the partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$ must exist at point (x, y).

Similarly, *if we take Δz to be wholly imaginary so that $\Delta x = 0$* ; then we get

$$f'(z) = \lim_{\Delta y \to 0} \left[\frac{\{u(x, y + \Delta y) + iv(x, y + \Delta y)\} - \{u(x, y) + iv(x, y)\}}{i\,\Delta y} \right]$$

$$= \lim_{\Delta y \to 0} \left[\frac{u(x, y + \Delta y) - u(x, y)}{i\, \Delta y}\right] + i \lim_{\Delta y \to 0} \left[\frac{v(x, y + \Delta y) - v(x, y)}{i\, \Delta y}\right]$$

$$= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \qquad \text{... (2)}$$

Since $f'(z)$ exists, the derivatives $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$ must exist at point (x, y). As the function $f(z)$ is analytic, therefore, $f'(z)$ must approach a unique limit as $\Delta z \to 0$ *in any manner, i.e.,* the limits (1) and (2) must be identical. Thus, the necessary condition for the function $f(z)$ to be analytic is

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Equating real and imaginary parts, we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \qquad \text{... (3)}$$

These equations are called *Cauchy-Riemann differential equations. Thus the necessary condition for the function $f(z)$ to be analytic at the point z is that the four partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}$ should exist and satisfy the Cauchy-Riemann differential equations at point $z = (x, y)$.*

These results may be summed up as follows in the form of a theorem :

Theorem 6·2. *The real and imaginary parts of an analytic function $f(z) = u(x, y) + iv(x, y)$ satisfy the Cauchy-Riemann differential equations at each point where $f(z)$ is analytic.*

The Sufficient Conditions

Let $w = f(z) = u(x, y) + iv(x\ y)$. Let us assume that the real and imaginary parts $u(x, y)$ and $v(x, y)$ of two real variables x and y have continuous first order partial derivatives that satisfy the Cauchy-Riemann equations in some domain D.

Let $P : z = (x, y)$ be a point in the domain D. Let us choose a point $Q : z + \Delta z = (x + \Delta x, y + \Delta y)$ in the neighbourhood of z in the domain D. Then the segment PQ is in domain D and from the *mean value theorem* it follows that

$$\left.\begin{aligned} u(x + \Delta x, y + \Delta y) - u(x, y) &= \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y \\ v(x + \Delta x, y + \Delta y) - v(x, y) &= \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y \end{aligned}\right\} \qquad \text{... (2)}$$

Now, $f(z + \Delta z) - f(z)$

$$= [u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y)] - [u(x, y) + iv(x, y)]$$

$$= [u(x + \Delta x, y + \Delta y) - u(x, y)] + i[v(x + \Delta x, y + \Delta y) - v(x, y)]$$

$$= \left(\frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y\right) + i\left(\frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y\right)$$

Using Cauchy-Riemann equations (3) $\left(\text{i.e. } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}\right)$ we get

$$f(z + \Delta z) - f(z) = \left(\frac{\partial u}{\partial x}\Delta x - \frac{\partial v}{\partial x}\Delta y\right) + i\left(\frac{\partial v}{\partial x}\Delta x + \frac{\partial u}{\partial x}\Delta y\right)$$

$$= \frac{\partial u}{\partial x}(\Delta x + i\,\Delta y) + i\frac{\partial v}{\partial x}(\Delta x + i\,\Delta y)$$

$$= \left\{\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right\}(\Delta x + i\,\Delta y) = \left\{\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right\}\Delta z$$

Dividing both sides by Δz, we obtain

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

If we let $\Delta z \to 0$, then the derivatives on the right become $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$, evaluated at (x, y) ; hence the limit

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$ exists and we see that it is $f'(z)$.

This shows that $f(z)$ is analytic in the domain D.

Thus, *the sufficient conditions for the function $f(z) = u(x, y) + iv(x, y)$ to be analytic in some domain D are that the real functions $u(x, y)$ and $v(x, y)$ of real variables x and y should have contininous first order partial derivatives that satisfy the Cauchy-Riemann equations in domain D.*

These results may be summed up in the form of a theorem as follows :

Theorem 6·3. *If two real single-valued functions $u(x, y)$ and $v(x, y)$ of two variables x and y have continuous first order partial derivatives that satisfy the Cauchy-Riemann equations in some domain D, then the complex function $f(z) = u(x, y) + iv(x, y)$ is analytic in domain D.*

Thus the Cauchy-Riemann equations are fundamental because they are not only necessary but are also sufficient for a function to be analytic. The theorems 6·2 and 6·3 are of great practical importance and it may now be easily decided whether or not a given complex function is analytic.

Cauchy-Riemann equations in polar form

The Cauchy-Riemann equations in cartesian form are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \qquad \text{...(1)}$$

Let the polar coordinates of $z(x, y)$ be (r, θ). Then

$$x = r\cos\theta,\; y = r\sin\theta$$

So that $\quad r^2 = x^2 + y^2$ and $\theta = \tan^{-1}\frac{y}{x}$

Then $\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta$, $\frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta$

$$\frac{\partial \theta}{\partial x} = \left[\frac{1}{1 + \left(\frac{y^2}{x^2}\right)} \cdot \left(\frac{-y}{x^2}\right)\right] = -\frac{y}{x^2 + y^2} = -\frac{r\sin\theta}{r^2} = -\frac{\sin\theta}{r}$$

Similarly, $\frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}$

Therefore

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r} \qquad \text{... (2)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y} = \frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r} \qquad \text{... (3)}$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial v}{\partial r}\cos\theta - \frac{\partial v}{\partial \theta}\frac{\sin\theta}{r} \quad \text{... (4)}$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial y} = \frac{\partial v}{\partial r}\sin\theta + \frac{\partial v}{\partial \theta}\frac{\cos\theta}{r} \quad \text{... (5)}$$

Substituting these values in equation (1), we get

$$\frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r} = \frac{\partial v}{\partial \theta}\sin\theta + \frac{\partial v}{\partial \theta}\frac{\cos\theta}{r} \quad \text{... (6)}$$

and
$$\frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r} = -\left(\frac{\partial v}{\partial r}\cos\theta - \frac{\partial v}{\partial \theta}\frac{\sin\theta}{r}\right) \quad \text{... (7)}$$

Multiplying (6) by $\cos\theta$ and (7) by $\sin\theta$ and adding, we get

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{... (8)}$$

Again multiplying (6) by $(-\sin\theta)$ and (7) by $\cos\theta$ and adding ; we get

$$\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r} \quad \text{... (9)}$$

Equation (8) and (9) represent Cauchy-Riemann equations in polar form.

6·11. Laplace's Equation ; Harmonic Functions

It will be proved later that the *derivative of an analytic function* $f(z) = u(x, y) + iv(x, y)$ *is itself analytic*. By this important fact $u(x, y)$ and $v(x, y)$ will have continuous partial derivatives of all orders. In particular, the mixed second order derivatives of these functions will be given by

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x} \text{ and } \frac{\partial^2 v}{\partial x\,\partial y} = \frac{\partial^2 v}{\partial y\,\partial x}$$

Let $f(z) = u(x, y) + iv(x, y)$ be an analytic function ; then Cauchy-Riemann equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Differentiating these equations ; we obtain

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x\,\partial y} \;;\; \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y\,\partial x} \quad \text{... (1)}$$

and
$$\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x\,\partial y} \;;\; \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial x\,\partial y} \quad \text{... (2)}$$

Adding equations (1) and (2) successively, we get

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{... (3)}$$

$$\nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \quad \text{... (4)}$$

These equations are called Laplace's equations (in two dimensions) whose solutions are the functions $u(x, y)$ and $v(x, y)$ with the condition that their second order partial derivatives exist. Hence we arrive at the following important theorem :

Theorem 6· 4. *The real and imaginary parts of a complex function* $f(z) = u(x, y) + iv(x, y)$ *that is analytic in a domain D have continuous second order partial derivatives and are the solutions of Laplace's equations*

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \;;\; \nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

A solution of Laplace's equation having continuous second order partial derivatives is called a *harmonic function*. If the function $f(z) = u(x, y) + iv(x, y)$ is analytic in domain D; then $v(x, y)$ is said to be the *conjugate harmonic function* of $u(x, y)$ and the functions $u(x, y)$ and $v(x, y)$ are called *conjugate harmonic functions* in domain D (of course this is a different use of word *conjugate* from that employed in defining $\bar{z}$, the conjugate of a complex number z).

SOLVED EXAMPLES

Ex.6. *Which of the following are analytic functions of complex variable $z = x + iy$;*

(*i*) $|z|$ (*ii*) $R_e\, z$ (*iii*) z^{-1}

(*iv*) $\sin z$ (*v*) $e^{\sin z}$.

(*Madras 2004, Rohilkhand 2008, 1997, 94*)

Solution. The necessary and sufficient conditions for a function $f(z) = u + iv$ to be analytic are that

(*a*) Both real single-valued functions u and v must have continuous first order partial derivatives and

(*b*) The Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

must be satisfied.

(*i*) We have $z = x + iy$; therefore $|z| = |x + iy| = \sqrt{x^2 + y^2}$

Given $f(z) = u + iv = |z| = \sqrt{x^2 + y^2}$

Comparing real and imaginary parts

$$u = \sqrt{x^2 + y^2} \text{ and } v = 0$$

Obviously, the first order partial derivatives of v *i.e.*, $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ do not exist. Hence the function $f(z) = |z|$ is *not analytic*.

(*ii*) Given $f(z) = R_e z = R_e(x + iy) = x$

i.e., $u + iv = x$

Thus we have $u = x$ and $v = 0$

Obviously the first order partial derivtatives $\frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ do not exist ; hence the function $f(z) = R_\theta(z)$ is *not analytic*.

(*iii*) Given $f(z) = z^{-1} = \frac{1}{z} = \frac{1}{x + iy} = \frac{x - iy}{(x + iy)(x - iy)}$

i.e., $f(z) = u + iv = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}$

Comparing real and imaginary parts, we get

$$u = \frac{x}{x^2 + y^2} = x(x^2 + y^2)^{-1} \text{ and } y = -\frac{y}{x^2 + y^2} = -y(x^2 + y^2)^{-1}$$

Taking partial derivatives of u and v, we have

$$\frac{\partial u}{\partial x} = -\frac{(x^2 - y^2)}{(x^2 + y^2)^2} ; \frac{\partial u}{\partial y} = -\frac{2xy}{(x^2 + y^2)^2}$$