$$= -\frac{1}{r}\left(-3r^3 \cos 3\theta\right) dr + r(-3r^2 \sin 3\theta)\, d\theta$$

$$= 3r^2 \cos 3\theta\, dr - 3r^3 \sin 3\theta\, d\theta = d\,(r^3 \cos 3\theta)$$

$\Rightarrow \quad v = r^3 \cos 3\theta$

$\therefore \quad f(z) = 4 + iv = -r^3 \sin 3\theta + ir^3 \cos 3\theta$

$$= ir^3 (\cos 3\theta + i \sin 3\theta)$$

$$= ir^3\, e^{3i\theta} = i\,(re^{2\theta})^3$$

$\Rightarrow \quad f(z) = iz^3$

6·12. Line Integral of a Complex Function

Let us define the line integral (or definite integral) of a complex function $f(z)$ where $z = x + iy$ is complex variable defined in the complex plane.

Let C be the smooth curve in the complex z-plane having end points z_0 and z_n. Let $z_1, z_2, z_3 .. z_{r-1}, z_r, ... z_{n-1}$ be an arbitrary number of intermediate points which subdivide the curve C into n arcs $z_0 z_1, z_1 z_2, z_2 z_3 ... z_{r-1} z_r, ... z_{n-1} z_n$ (Fig. 6·5). Let $\xi_1, \xi_2, \xi_3 ... \xi_r, ..., \xi_n$ be the points chosen on the curve C such that ξ_1 lies on arc $z_0 z_1$, ξ_2 lies on arc $z_1 z_2$, ... ξ_r lies on are $z_{r-1} z_r$ and so on.

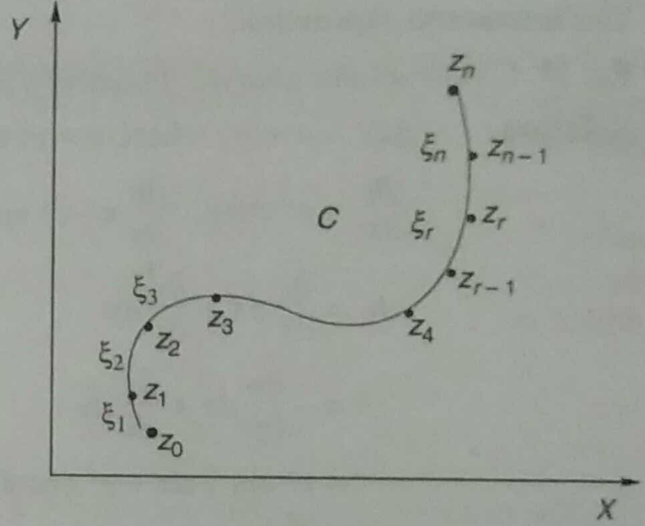


Fig. 6·5

Let the following summation be performed.

$$S_n = \sum_{r=1}^{n} f(\xi_r)\, \Delta z_r \qquad \text{... (1)}$$

where $\quad \Delta z_r = z_r - z_{r-1}$

If the curve C is divided into smaller and smaller parts so that $n \to \infty$, $|\Delta z_r| \to 0$ and if the summation tends to a *unique limit* that is independent of the choice of intermediate points and of the manner in which the subdivision is performed, then the unique limit S_n is called the *line integral* (or the definite integral) of the complex function $f(z)$ taken along the curve C between the points z_0 and z_n and is denoted by

$$\lim_{n \to \infty} S_n = \int_{z_0,\, C}^{z_n} f(z)\, dz \qquad \text{... (2)}$$

This definition of complex line integral as the limit of a sum is a natural generalization of a familiar definition of a real definite integral.

Rectifiable Curve. Let C be smooth curve in the complex z-plane. The curve C may be represented in the form

$$z = z(t) = x(t) + iy(t) \qquad \text{... (3)}$$

where t takes values between α and β *i.e.*, $\alpha \le t \le \beta$.

Let the interval (α, β) be subdivided into sub-intervals by the points $t_0 (= \alpha), t_1, t_2, ... t_r ... t_{n-1}, t_n (=\beta)$ and let these points on the curve be denoted by $z_0, z_1, z_2, z_2 ... z_{n-1}, z_n (= z)$.

Join z_0 to z_1, z_1 to z_2, z_2 to z_3 ... z_{n-1} to z_n by straight lines.

Then the length of the polygonal line $z_0 z_1 z_2 \ldots z_n$ is given by

$$L = \text{length } z_0 z_1 + \text{length } z_1 z_2 + \ldots + \text{length } z_{n-1} z_n$$

$$= [x_1 - x_0)^2 + (y_1 - y_0)^2]^{1/2} + [x_2 - x_1)^2 + (y_2 - y_1)^2]^{1/2} + \ldots + [(x_n - x_{n-1})^2 + (y_n - y_{n-1})^2]^{1/2}$$

$$= \sum_{r=1}^{n} [(x_r - x_{r-1})^2 + (y_r - y_{r-1})^2]^{1/2}$$

$$= \sum_{r=1}^{n} | z_r - z_{r-1} | = \sum_{r=1}^{n} | \Delta z_r | \quad \ldots (4)$$

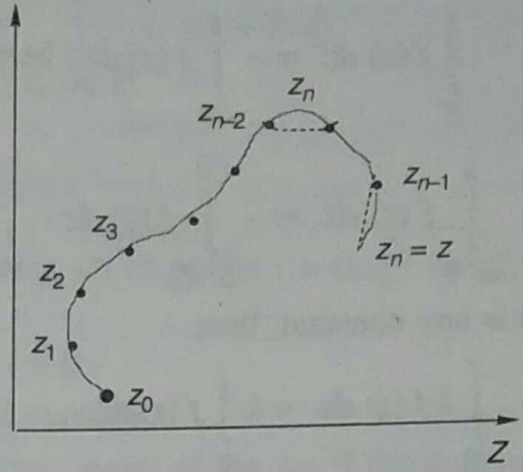


Fig. 6·6

The value of this sum depends on the particular mode of subdivision of (α, β) and is called the *length of an inscribed polygon L*. If the curve is such that the length of all the inscribed polygons have a finite upper bound l, then the curve is said to be rectifiable and the length of the curve is taken as l.

In equation (4) $| \Delta z_r | = | z_r - z_{r-1} |$ is the length of the chord whose end points are z_{r-1} and z_r. If n approaches infinity such that the greatest $| \Delta z_r |$ approaches zero, then L approaches the finite upper bound l, the length of the curve C. From this the length of the curve C is defined as

$$l = \lim_{n \to \infty} \sum_{r=1}^{n} | \Delta z_r | = \int_{z_0, C}^{z} | dz | \quad \ldots (5)$$

The necessary and sufficient condition for the curve C to be rectifiable is that the functions $x(t)$, $y(t)$ should be of bounded variation in the interval (α, β). In case the first derivatives of $x(t)$, $y(t)$ are continuous, the curve

$$z(t) = x(t) + iy(t) \text{ where } \alpha \le t \le \beta$$

is rectifiable and its length l is denoted by

$$l = \int_{\alpha}^{\beta} [\{\dot{x}(t)\}^2 + \{\dot{y}(t)\}^2]^{1/2} \, dt \quad \ldots (6)$$

where $\dot{x}(t) = \frac{dx}{dt}$ and $\dot{y}(t) = \frac{dy}{dt}$

Basic Properties of the Complex Line Integrals. From the definition of a complex line integral as the limit of a sum we immediately obtain the following properties of complex integrals :

1. If $f_1(z)$ and $f_2(z)$ are continuous complex functions over a continuous rectifiable curve C, so that $f_1(z)$, $f_2(z)$ and $f_1(z) + f_2(z)$ are all integrable over C, then

$$\int_C [f_1(z) + f_2(z)] \, dz = \int_C f_1(z) \, dz + \int_C f_2(z) \, dz \quad \ldots (7)$$

2. If $f(z)$ is continuous complex function over a continuous rectifiable curve C and C_1, C_2 are two decomposed parts of C, then

$$\int f(z) \, (dz) = \int f(z) \, dz + \int_{C_2} f(z) \, dz \quad \ldots (8)$$

3. If we reverse the sense of integration, the sign of the integral value changes, *i.e.*,

$$\int_C f(z)\,dz = -\int_{-C} f(z)\,dz \text{ ; here } -C \text{ represents the sense opposite to that of } C.$$

or

$$\int_{z_0, C}^{z} f(z)\,dz = -\int_{z, C}^{z_0} f(z)\,dz \qquad \dots (9)$$

4. If k is any constant, then

$$\int_C k f(z)\,dz = k\int_C f(z)\,dz \qquad \dots (10)$$

Estimation of absolute value of complex line integral

Let C be a rectifiable curve (*i.e.*, of finite length) ; then

$$\left|\int_C f(z)\,dz\right| \le Ml \qquad \dots (11)$$

where l is the length of the curve and M is a real number such that

$$|f(z)| \le M \text{ for every } z \text{ on } C. \qquad \dots (12)$$

We have $\quad S_n = \sum_{r=1}^{n} f(\xi_r)\,\Delta z_r$

As the modulus of the sum of n complex numbers can never exceed the sum of their moduli, therefore

$$\left|S_n\right| \le \sum_{r=1}^{n}\left|f(\xi_r)\,\Delta z_r\right| \le \sum_{r=1}^{n} |f(\xi_r)|\,|\Delta z_r|$$

For $n \to \infty$, this inequality can be written as

$$\left|\int_C f(z)\,dz\right| \le \int_C |f(z)|\;|dz| \le M\int_C |dz| \qquad \dots (13)$$

using (12)

But $\int_C |dz| = \lim_{n\to\infty}\sum_{r=1}^{n}|\Delta z_r| = \lim_{n\to\infty}\sum_{r=1}^{n}|z_r - z_{r-1}| = l$, the length of the curve C.

Equation (13) gives

$$\left|\int_C f(z)\,dz\right| \le Ml$$

Reduction of $\int_C f(z)\,dz$ to Real Integrals. Let us decompose $f(z)$ and z into real and imaginary parts ; then

$$f(z) = u + iv \text{ and } dz = dx + i\,dy$$

So that $\int_{z_0, C}^{z} f(z)\, dz = \int_{x_0, y_0, C}^{x, y} (u\, dx - v\, dy) + i \int_{x_0, y_0, C}^{x, y} (v\, dx + u\, dy)$ ... (14)

where on R.H.S. we have two real line integrals.

6·13. A Few Preliminary Concepts

Continuous Arc. A continuous arc is defined as a set of points $z = (x, y)$ satisfying the equation

$$z = x(t) + i\, y(t) \text{ in the range } \alpha \le t \le \beta$$

where $x(t)$ and $y(t)$ are continuous functions of the real variable t.

Multiple Points. A point z_1 is said to be a *multiple point* of the arc if the equation $z_1 = x(t) + i\, y(t)$ is satisfied by two more values of t in the given range.

Jordan Arc. A continuous arc having no multiple points is called a Jordan arc.

Jordan Curve or Simple Closed Curve. Consider a continuous arc defined by

$$z = x(t) + i\, y(t) \text{ in the range } \alpha \le t \le \beta \quad \text{... (1)}$$

If $x(\alpha) = x(\beta)$ and $y(\alpha) = y(\beta)$ and if there is no other multiple point on the continuous arc defined by (1), then this continuous arc is said to be a Jordan curve or a simple closed curve.

The circle

$$x = a \cos t,\ y = a \sin t \quad (0 \le t \le 2\pi)$$

in an example of a simple closed curve.

Regular or Smooth Jordan Arc. A continuous arc defined by this equation

$$z = x(t) + i\, y(t) \text{ in the range } \alpha \le t \le \beta$$

is said to be smooth or regular Jordan arc if $x(t)$ and $y(t)$ are single-valued continuous functions of variable t and have continuous first order derivatives $\dot{x}$ and $\dot{y}$ which do not vanish simultaneously for any value of t.

Contour. A contour is a curve composed of a chain of a finite number of regular Jordan arcs. It may be open or closed. If the contour is closed and does not intersect itself, it is piecewise smooth Jordan curve called the closed contour. The boundaries of triangles and rectangles are examples of closed contour. The length of a contour is the sum of lengths of smooth arcs. Clearly, a contour is rectifiable.

Simply Connected and Multiply Connected Domains

A domain is simply connected if the interior of every closed contour in the domain consists only the points of the domain. A domain that is not simply connected is said to be multiply connected. The interior of a closed contour (e.g. circle, square, ellipse etc.) is simply connected while the annular region between two concentric circles is multiply connected (more precisely : doubly connected). In general, a domain is said to be m-fold connected if its boundaries consist

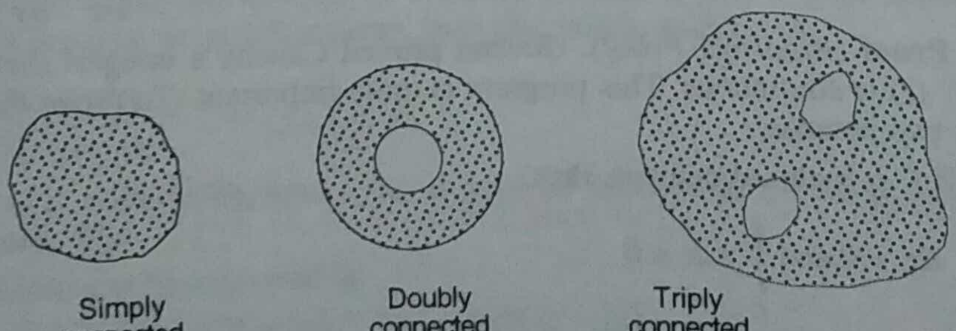


Fig. 6·7

of m distinct parts without common points. For the annular region between two concentric circles (or annulus), $m = 2$ because its boundary consists of two distinct circles having no points in common.

6·14. Cauchy's Integral Theorem

Theorem 6·5. *If $f(z)$ is analytic throughout a simply connected bounded domain D, then for every closed contour C in domain D*

$$\int_C f(z)\, dz = 0 \qquad \ldots (1)$$

Cauchy's Proof. The Cauchy's proof of Cauchy's integral theorem is elementary. It depends on the two-dimensional form of Stoke's theorem and requires the *additional assumption that* $f'(z)$ [first derivative of $f(z)$] *is continuous.*

If **A** is a vector field whose components possess the continuous partial derivatives involved in the calculation of $\nabla \times \mathbf{A}$ in domain D, then Stoke's theorem states

$$\int_C \mathbf{A} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S}, \qquad \ldots (2)$$

where the surface integration extends over the surface bounded by closed contour C. If **A** is two-dimensional vector field having the components A_x and A_y, then eqn. (2) reduces to

$$\int_C (A_x\, dx + A_y dy) = \iint_S \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) dx\, dy \qquad \ldots (3)$$

where C is a closed contour lying in the x-y plane and S is the surface bounded by this contour.

Decomposing $f(z)$ and z into real and imaginary parts, *i.e.*

$$f(z) = u + iv,\ z = x + iy$$

we get

$$\int_C f(z)\, dz = \int_C (u + iv)(dx + i\, dy) = \int_C (u\, dx - v\, dy) + i \int_C (v\, dx + u\, dy) \qquad \ldots (4)$$

Transforming both the integrals on right hand side of (4) into surface integrals by the use of (3), we get

$$\int_C f(z)\, dz = \iint_S \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\, dy + i \iint_S \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\, dy$$

$$= 0 \text{ by virtue of Cauchy-Riemann equations } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Goursat's Proof (*Rigorous Proof*). Goursat proved Cauchy's integral theorem without assuming that $f'(z)$ is continuous. This progress is quite important. To prove the theorem let us first consider two lemmas.

Lemma I. If C is a closed contour, then

$$\int_C dz = 0 \text{ and } \int_C z\, dz = 0 \qquad \ldots (5)$$

It follows from definition of complex integral of $f(z)$

$$\int_C f(z)\,dz = \lim_{n\to\infty}\left[\sum_{r=1}^{n} f(\xi_r)(z_r - z_{r-1})\right]$$

where ξ_r is any point on the arc $z_r z_{r-1}$.

Taking $f(z) = 1$; we get

$$\int_C dz = \lim_{n\to\infty}\sum_{r=1}^{n}[1 \cdot (z_r - z_{r-1})] = (z_1 - z_0) + (z_2 - z_1) + \ldots + (z_n - z_{n-1})$$

$$= (z_n - z_0) = 0 \qquad \text{(since } z_n \text{ and } z_0 \text{ coincide for a closed contour)}$$

Taking $f(z) = z$ we get

$$\int_C z\,dz = \lim_{n\to\infty}\left[\sum_{r=1}^{n}(\xi_r)(z_r - z_{r-1})\right]$$

$$= \lim_{n\to\infty}\left[\sum_{r=1}^{n} z_r (z_r - z_{r-1})\right] \text{ considering } \xi_r \text{ to be the point } z_r \qquad \ldots (6)$$

$$\text{Also} \int_C z\,dz = \lim_{n\to\infty}\left[\sum_{r=1}^{n} z_{r-1}(z_r - z_{r-1})\right] \text{ considering } \xi_r \text{ to be the point } z_{r-1} \qquad \ldots (7)$$

Adding (6) and (7), we get

$$\int_C z\,dz = \frac{1}{2}\lim_{n\to\infty}\left[\sum_{r=1}^{n}\{z_r(z_r - z_{r-1}) + z_{r-1}(z_r - z_{r-1})\}\right]$$

$$= \frac{1}{2}\lim_{n\to\infty}\left[\sum_{r=1}^{n}(z_r^2 - z^2_{r-1})\right] = \frac{1}{2}\lim_{n\to\infty}(z_n^2 - z^2_0)$$

$= 0$ since z_n and z_0 coincide for a closed contour.

Lemma II : Goursat's Lemma. Given any positive number $\in$, however small, it is always possible to divide the interior of a closed contour C into a finite number of meshes, either complete squares or partial squares, such that within each mesh

$$\left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| < \in \qquad \ldots (8a)$$

for all values of z in the mesh, where $f(z)$ is analytic function of z at all points within and on the closed contour C.

Condition (8a) can be expressed as

$$f(z) = f(z_0) + (z - z_0) f'(z_0) + \eta(z)(z - z_0) \qquad \ldots (8b)$$

for all values of z in the mesh, where $|\eta(z)| < \in$.

To prove this lemma let us consider a closed contour C in a simply connected bounded domain D where $f(z)$ is analytic throughout. Let the interior of the contour C be subdivided into a finite number of meshes out of which some are complete squares while some are partial squares (Fig. 6·8 *a*).

Let us further suppose that the lemma is false; then however the inteiror of C is subdivided into meshes, there will be at least one mesh for which condition (8) is untrue. Let this particular mesh be denoted by T_1. Now the mesh T_1 is quadrisected by lines joining midpoints of the opposite sides ; then there is at least one of the four quarters of T_1 for which condition (8) is not satisfied. Let it be denoted by T_2. Again T_2 is quadrisected and its quarter T_3 (say) is taken for which condition (8) does not hold. This process is carried on indefinitely. Thus we get an infinite sequence of meshes $T_1, T_2, T_3 \ldots T_n \ldots$, each contained in the preceding one for which the lemma is untrue. This infinite sequence of meshes determines a limit point ξ (say) which lies within the closed contour C and does not satisfy condition (8). As equation (8) is not being satisfied at ξ, therefore, we must have

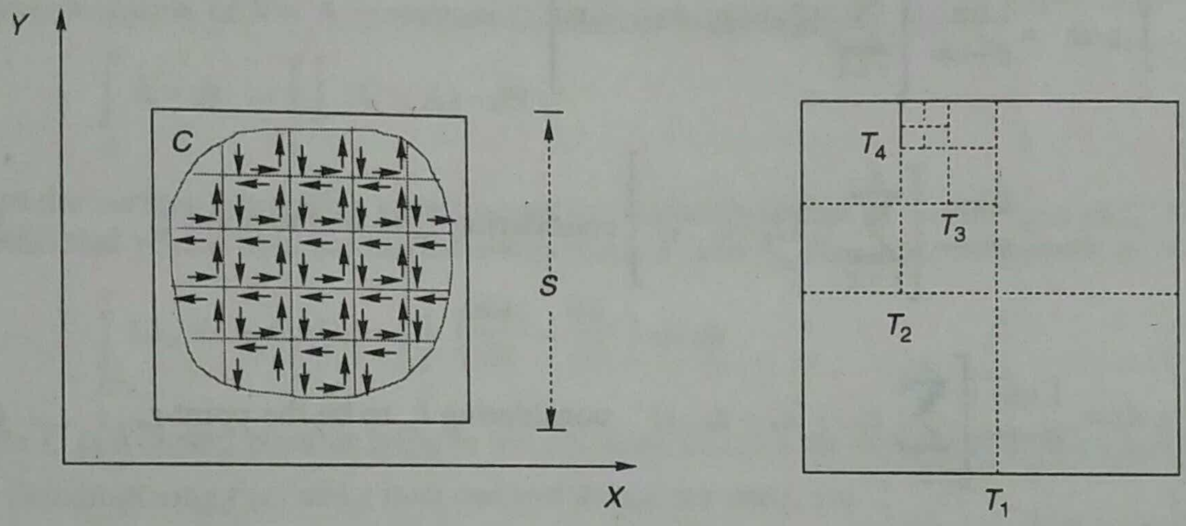


Fig. 6·8

$$\left| \frac{f(z) - f(\xi)}{z - \xi} - f'(\xi) \right| \nless \epsilon \text{ where } |z - \xi| < \delta$$

This relation shows that $f(z)$ is not analytic at the point ξ which contradicts our hypothesis that $f(z)$ is analytic at all points within and on the closed contour C. In other words, in accordance to our hypothesis ξ must satisfy condition (8). This proves Goursat's lemma.

Proof of the Theorem. Let the interior of the contour C be subdivided into a finite number of meshes, squares and partial squares, having boundaries $C_1, C_2, C_3, \ldots C_j \ldots C_n$. Let Goursat's lemma be true in each of these meshes.

Let the integral around each C_j be taken in the counterclockwise sense. The sum of all these integrals is the integral around the closed contour C in the counterclockwise sense (Fig. 6·8 *a*) ; *i.e.*,

$$\int_C f(z)\, dz = \sum_{j=1}^{n} \int_{C_j} f(z)\, dz \qquad \ldots (9)$$

because the line integrals along the common boundary line of every pair of adjacent meshes cancel each other ; the integral is taken in one sense along that line in one mesh and in the opposite sense in the other ; hence only the integrals along the arcs that are parts of C remain.

If z_j is point in th mesh C_j for which Goursat's lemma is true, then we have

$$\left| \frac{f(z) - f(z_j)}{z - z_j} f'(z_j) \right| < \epsilon$$

which can be written as

$$f(z) = f(z_j) + (z - z_j) f'(z_j) + (z - z_j)\,\eta(z) \text{ where } |\eta(z)| < \epsilon$$

Hence $\int_{C_j} f(z)\,dz = \int_{C_j} [f(z_j) + (z - z_j) f'(z_j) + (z - z_j)\,\eta(z)]\,dz$

$$= \left\{ f(z_j) - z_j f'(z_j) \right\} \int_{C_j} dz + f'(z_j) \int_{C_j} z\,dz + \int_{C_j} (z - z_j)\,\eta(z)\,dz$$

But $\int_{C_j} dz = 0$ and $\int_{C_j} z\,dz = 0$ according to lemma I; therefore

$$\int_{C_j} f(z)\,dz = \int_{C_j} (z - z_j)\,\eta(z)\,dz \qquad \text{... (10)}$$

Therefore in view of equation (9)

$$\int_C f(z)\,dz = \sum_{j=1}^{n} \int_C (z - z_j)\,\eta(z)\,dz$$

and hence $\left| \int_C f(z)\,dz \right| \le \sum_{j=1}^{n} \left| \int_{C_j} (z - z_j)\,\eta\,dz \right| \le \sum_{j=1}^{n} \int_{C_j} |z - z_i|\,|\eta(z)|\,|dz|$

$$< \epsilon \sum_{j=1}^{n} \int_{C_j} |z - z_j|\,|dz| \qquad \text{... (11)}$$

Since $|\eta(z)| < \epsilon$

Each boundary C_j coincides either entirely or partially with the boundary of a square. In either case let S_j denote the length of the side of that square. Now z is on the boundary C_j and z_j is either within or on C_j, so that

$$|z - z_j| \le l_j \sqrt{2} \text{ (= length of the diagonal of } j^{th} \text{ square)}$$

$$\therefore \quad \int_{C_j} |z - z_j|\,|dz| \le l_j \sqrt{2} \int_{C_j} |dz| \qquad \text{... (12)}$$

The integral $\int_{C_j} |dz|$ represents the length of mesh C_j. It is $4\,l_j$ if C_j is a square and it does not exceed $(4l_j + L_j)$ if C_j is a partial square, where L_j is the arc of the contour C that forms a part of C_j.

When C_j is square and A_j $(= l_j^2)$ denotes the area of that square, then according to inequality (12)

$$\int_{C_j} |z - z_j|\,|dz| \le l_j \sqrt{2}\cdot 4l_j = 4\sqrt{2}\, l_j^2 = 4\sqrt{2}\, A_j \qquad \text{... (13)}$$

When C_j is a partial square, then

$$\int_{C_j} |z - z_j|\,|dz| < l_j \sqrt{2}\,(4l_j + L_j) = (4\sqrt{2}\, l_j^2 + \sqrt{2}\, l_j L_j) < (4\sqrt{2}\, A_j + \sqrt{2}\, SL_j) \qquad \text{... (14)}$$

where S is the length of a side of some square that encloses the entire curve C as well as all squares used originally in covering C (Fig. 6·8*a*). Thus the sum of all A_j cannot exceed S^2.

If L denotes the length of contour C, then it follows from the inequalities (11), (13) and (14) that

$$\left| \int_C f(z)\, dz \right| < \in (4\sqrt{2}\, S^2 + \sqrt{2}\, SL)$$

But $\in$ is an arbitrary small positive number, therefore by choosing $\in$ (>0) sufficiently small we can make the expression on the right as small as we please, while the expression on the left is a definite value of an integral. From this we conclude that this value must be zero *i.e.*,

$$\int_C f(z)\, dz = 0$$

This proves Cauchy's integral theorem.

Extension to multiply connected domains. Cauchy's integral theorem has been deduced under the assumption that the closed contour C is boundary of a simply connected domain. However, the theorem can be extended to hold for certain multiply connected domains whcih can be cut such that the resulting domain becomes simply connected.

Consider the multiply connected region of Fig. 6·9 in which $f(z)$ is not defined for the interior regions R_1 and R_2 and the region exterior to R. Let us draw a closed contour C in the multiply connected region R. Cauchy's integral theorem is not valid for the countour C, as shown but we can construct a contour C' for which the theorem holds. By introducing the cross cuts $a_1 a_1'$, $b_1 b_1'$ (connecting forbidden region R_1 to the forbidden region exterior to R) and $a_2 a_2'$, $b_2 b_2'$ (connecting forbidden region R_2 to the forbidden region exterior to R) the contour C'

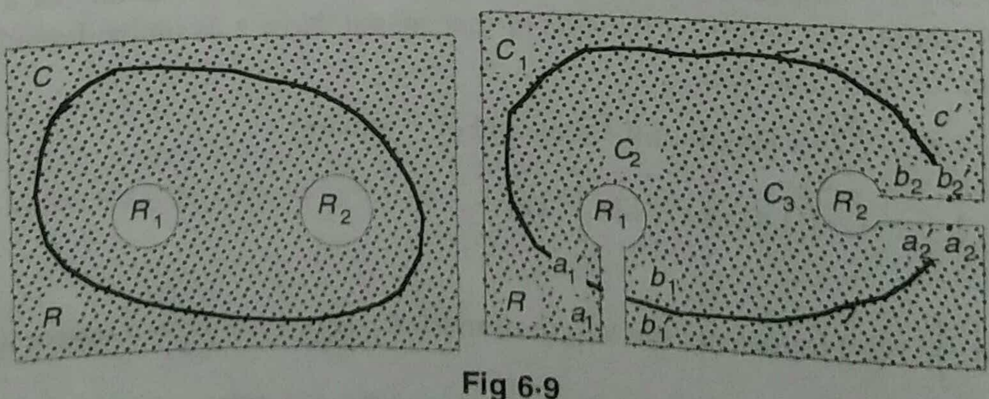


Fig 6·9

(through $C_1 a_1 a_1' C_2 b_1 b_1' a_2 a_2' C_3 b_2 b_2' C' C_1$) becomes the boundary of simply connected domain. This literally transforms the multiply connected region R into simply connected one. Therefore it follows from Cauchy's integral theorem that the integral of $f(z)$ taken over the boundary of C' in the sense indicated by arrows in Fig. 6·9 *b* has the value zero *i.e.*,

$$\int_{C'} f(z)\, dz = 0$$

Since the function $f(z)$ is analytic along the cross cuts $a_1a'_1$, $b_1b'_1$, and $a_2a'_2$, $b_2b'_2$ and integral along cross cuts is taken in both directions therefore the integrals along cross cuts cancel out in pairs and hence we obtain

$$\int_{C'} f(z)\, dz = \int_{C_1} f(z)\, dz + \int_{C_2} f(z)\, dz + \int_{C_3} f(z)\, dz = 0 \qquad \text{... (15)}$$

where C_1 is traversed in counterclockwise sense and C_2, C_3 are traversed in the opposite sense.

If all the curves C_1, C_2, C_3 are traversed in the same sense, then

$$\int_{C_1} f(z)\, dz = \int_{C_2} f(z)\, dz + \int_{C_3} f(z)\, dz \qquad \text{... (16)}$$

The process of transforming multiply connected regions to simply connected regions is applicable even to more complicated regions.

Ex. 21. *Use Cauchy's integral theorem to evaluate the integral*

$$\oint_C \frac{dz}{z}$$

where *C is a simple closed curve*

Solution. The function $f(z) = \dfrac{1}{z}$ is analytic for every value of z except for $z = 0$. Therefore if closed contour C encloses the origin, we draw an arc C_1 of small radius r with centre at the origin (Fig. 6·10). As $f(z)$ is analytic in the region between C_1 and C, therefore from equation (16) we have

$$\oint_C \frac{dz}{z} = \oint_{C_1} \frac{dz}{z} \qquad \text{... (1)}$$

On contour C_1, we have

$$z = re^{i\theta},\ dz = ire^{i\theta}\, d\theta$$

Hence $$\oint_{C_1} \frac{dz}{z} = \int_0^{2\pi} i\,d\theta = 2\pi i$$

Hence from (*i*)

$$\oint_C \frac{dz}{z} = 2\pi i$$

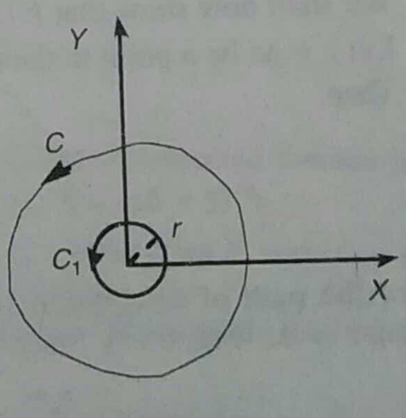


Fig. 6·10

However, if closed contour C does not enclose the origin, then $f(z) = 1/z$ is analytic throughout on and within C, hence by Cauchy's integral theorem

$$\oint_C \frac{dz}{z} = 0$$

Thus we have the result

$$\oint_C \frac{dz}{z} = \begin{cases} 0 \text{ if } C \text{ does not enclose origin} \\ 2\pi i \text{ if } C \text{ encloses origin} \end{cases}$$

6·15. Evaluation of Line Integrals by Indefinite Integration

As a consequence of Cauchy's integral theorem complex line integrals in many cases may be evaluated by a very simple method, namely, by indefinite integration.

A function $F(z)$ is said to be an indefinite integral of $f(z)$ in a domain D if

$$F'(z) = f(z) \text{ for all points } z \text{ in } D$$

Let z_0 and z represent two points in a simply connected domain D throughout which $f(z)$ is an analytic function of z. Let C_1 and C_2 be any two continuous rectifiable curves connecting z_0 to z and lying entirely within D; then C_1 and C_2 together form a closed contour C (say) along which Cauchy's integral theorem is applicable.

By Cauchy's theorem

$$\oint_C f(z)\,dz = \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = 0$$

i.e.,

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

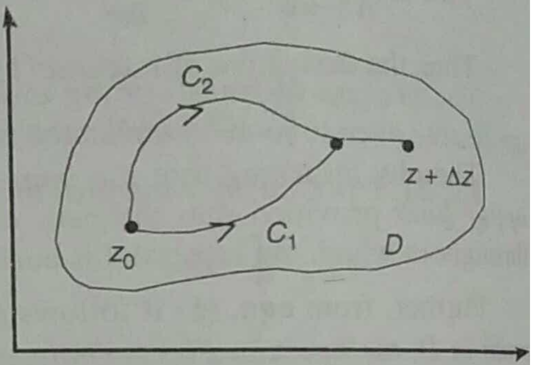


Fig. 6·11

This shows that the integral along any curve joining z_0 to z is independent of the path in D: but depends only on its end points.

In other words, the integral of any analytic function in the complex plane is independent path between given end points.

Thus we may write

$$F(z) = \int_{z_0}^{z} f(\xi)\,d\xi \qquad \text{...(}$$

where ξ is a variable of integration.

We shall now show that $F(z)$ *is an analytic function of* z *in* D *and that* $F'(z) = f(z)$.

Let $z + \Delta z$ be a point in the neighbourhood of z in the domain D.

Then

$$F(z + \Delta z) - F(z) = \int_{z_0}^{z+\Delta z} f(\xi)\,d\xi - \int_{z_0}^{z} f(\xi)\,d\xi = \int_{z}^{z+\Delta z} f(\xi)\,d\xi$$

where the path of integration from z to $z + \Delta z$ may be chosen as a straight line segm joining $z + \Delta z$ (Fig. 6·11). Since for fixed z we can write

$$f(z) = \frac{f(z)}{\Delta z}\int_{z}^{z+\Delta z} d\xi = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(z)\,d\xi$$

Hence

$$\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} [f(\xi) + f(z)]\,d\xi$$

But $f(z)$ is continuous function of z. Hence for each positive real number $\in$, there exists a positive number δ such that

$[f(\xi) - f(z)] < \in$ where $|\xi - z| < \delta$ or in particular when $|\Delta z| < \delta$. Consequently when $|\Delta z| < \delta$, then

$$\left| \frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) \right| = \frac{1}{|\Delta z|} \left| \int_z^{z+\Delta z} [f(\xi) - f(z)]\, d\xi \right|$$

$$< \left| \frac{\in}{|\Delta z|} \right| \int_z^{z+\Delta z} d\xi = \left| \frac{\in}{|\Delta z|} \right| |\Delta z| = \in$$

This is $\lim_{\Delta z \to 0} \frac{F(z+\Delta z) - F(z)}{\Delta z} = f(z)$.

Thus the derivative of integral (1) exists at each point z in D, and

$$F'(z) = f(z) \qquad \ldots (1)$$

Thereby indicatng that *the integral of an analytic function is an analytic function of its upper limit* provided that the path of integration is confined to a simply connected region throughout which the integrand is analytic.

Further, from eqn. (1) it follows *that when the lower limit z_0 is replaced by another fixed point in D, the function F(z) is changed by an additive constant.* Therefore $F(z)$ is an *indefinite integral* or *antiderivative* of $f(z)$ written as

$$F(z) = \int f(z)\, dz$$

that is, $F(z)$ is an analytic function in D whose derivative is $f(z)$.

In veiw of equation (1) the complex line integrals can be evaluated as *the change in the value of the indefinite integral* as in the case of real integrals ; since for any path joining any two points z_1 and z_2 in D

$$\int_{z_1}^{z_2} f(z)\, dz = \int_{z_0}^{z_2} f(z)\, dz - \int_{z_0}^{z_1} f(z)\, dz = F(z_2) - F(z_1)$$

It is assumed that the paths of integration are confined to a simply connected domain in which $f(z)$ is analytic.

It should be noted that if G is any analytic function of z other than F such that $G'(z) = f(z)$, then $F'(z) - G'(z) = 0$ *i.e.*, the derivative of the function $\omega = F(z) - G(z)$ is zero, thereby indicating that the function $\omega = F(z) - G(z)$ is constant. These results may be summed up in the form of a theorem as follows.

Theorem 6·6. *If $f(z)$ is an analytic function in a simply connected domain D and if F(z) is an indefinite integral of f(z), then for all paths in D joining two points z_1 and z_2 in D,*

$$\int_{z_1}^{z_2} f(z)\, dz = F(z_2) - F(z_1)$$

This theorem enables up to evaluate complex line integrals by means of indefinite integration.

Ex. 22. *Evaluate* (*i*) $\int_i^1 (z+1)^2\,dz$ (*ii*) $\int_0^{\pi i} z\cos z^2\,dz$

Solution. (*i*) $\int_i^1 (z+1)^2\,dz = \left[\frac{(z+1)^3}{3}\right]_i^1 = \frac{2^3}{3} - \frac{(i+1)^3}{3} = \frac{8}{3} - \frac{(2i-2)}{3} = \frac{10-2i}{3}$

(*ii*) Let $I = \int_0^{\pi i} z\cos z^2\,dz$

Substituting $z^2 = t$, $2z\,dz = dt$, we get

$$I = \int_0^{-\pi^2} \frac{1}{2}\cos t\,dt = \frac{1}{2}\Big[\sin t\Big]_0^{-\pi^2} = -\frac{1}{2}\sin\pi^2$$

6·16. Cauchy's Integral Formula

The most important consequence of Cauchy's integral theorem is the Cauchy's integral formula. This formula and its conditions of validity may be stated as follows :

Theorem 6·7. *If $f(z)$ is analytic and single valued within and on a closed contour C and if z_0 is any point within C, then*

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz \text{ (Cauchy's integral formula)} \quad \text{... (1)}$$

where the integral is taken in the counterclockwise sense.

Proof : Although $f(z)$ is assumed analytic, the function $\phi(z) = \frac{f(z)}{z-z_0}$ is not analytic at the point $z = z_0$. Hence Cauchy's integral theorem is not applicable on the contour C for the function $\phi(z)$. Let us describe a small circle C_0 of radius r about z_0 which lies entirely within C. Now the function $\phi(z)$ is analytic in the region between C and C_0. By making a cross cut joining any point of C_0 to any point of C we form a closed contour C' within which $\phi(z)$ is analytic, so that Cauchy's integral theorem applies, *i.e,*

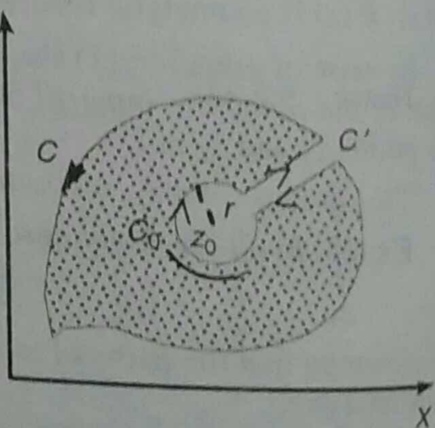


Fig. 6·12

$$\int_{C'} \phi(z)\,dz = 0 \text{ i.e. } \int_{C'} \frac{f(z)}{z-z_0}\,dz = 0$$

In traversing the contour C' in the counterclockwise sense the cross cut is traversed twice, once in each sense; hence it follows

$$\int_{C'} \frac{f(z)}{z-z_0}\,dz = \int_C \frac{f(z)}{z-z_0}\,dz - \int_{C_0} \frac{f(z)}{z-z_0}\,dz = 0$$

where all integrals are taken in counterclockwise sense.

Hence $$\int_C \frac{f(z)}{z-z_0}\,dz = \int_{C_0} \frac{f(z)}{z-z_0}\,dz \quad \text{... (2)}$$

Writing $f(z) = f(z_0) + [f(z) - f(z_0)]$ on right hand side and remembering that a constant may be taken out from under the integral sign, we get

$$\int_C \frac{f(z)}{z-z_0}\,dz = f(z_0)\int_{C_0} \frac{dz}{z-z_0} + \int_{C_0} \frac{f(z)-f(z_0)}{z-z_0}\,dz \quad \text{... (3)}$$

Now on C_0, $z - z_0 = re^{i\theta}$; $dz = re^{i\theta}\,d\theta$; so that

$$\oint_{C_0} \frac{dz}{z-z_0} = \oint_{C_0} \frac{ir\,e^{i\theta}\,d\theta}{r\,e^{i\theta}} = i\oint_{C_0} d\theta = 2\pi i \text{ for every positive } r \quad \text{... (4)}$$

Since $f(z)$ is analytic and therefore continuous at $z = z_0$; hence for every positive real number $\in$, however small, there exists a positive number δ such that

$|f(z) - f(z_0)| < \in$; so that

$$\left|\int_{C_0} \frac{f(z)-f(z_0)}{z-z_0}\,dz\right| \le \int_{C_0} \frac{|f(z)-f(z_0)|}{|z-z_0|}\,|dz|$$

$$< \int_{C_0} \frac{\in\,|ir\,e^{i\theta}\,d\theta|}{|r\,e^{i\theta}|} = \in \oint d\theta = 2\pi\in \quad \text{... (5)}$$

But in the limit $r = |z - z_0| \to 0$, $\in \to 0$ and since the other two integrals in equation (3) are independent of $\in$, in view of equation (4), hence the value of the integral in (5) must be equal to zero. Then equation (3) yields,

$$\int_C \frac{f(z)}{z-z_0}\,dz = f(z_0).\,2\pi i + 0$$

Hence $$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz$$

This proves Cauchy's integral formula.

Ex. 23. *Evaluate the integral*

$$\oint_C \frac{dz}{z^2+z}$$

(*Delhi 1999*)

where C is a circle defined by $|z| = |R > |$

Solution. The poles of integrand are given by putting denominator equal to zero i.e.,

$z^2 + z = 0 \Rightarrow z(z+1) = 0$ or $z = 0$ or $z = -1$

As radius of circle $|z| > 1$, therefore both these simple poles lie within the counter. Eliminating these poles by drawing circles of very small radii and making cross-cuts to form simply connected region, we get :

$$\oint_C \frac{dz}{z^2+z} = \oint_{C_1} \frac{dz}{z^2+z} + \oint_{C_2} \frac{dz}{z^2+z} = \oint_{C_1} \frac{dz/(z+1)}{z} + \oint_{C_2} \frac{(dz/z)}{z+1}$$

Using Cauchy integral formula $\int \frac{f(z)}{(z-z_0)} dz = 2\pi i f(z_0)$

$$\oint \frac{dz}{z^2+z} = 2\pi i\left[\frac{1}{z+1}\right]_{z=0} + 2\pi i\left[\frac{1}{z}\right]_{z=-1}$$

$$= 2\pi i - 2\pi i = 0$$

Ex. 24. *Find the value of* $\int_C \frac{z^2+1}{z^2-1} dz$, *if C is a circle of unit radius with centre at* $z = 1$.

Solution. Integrand $\frac{z^2+1}{z^2-1}$ has poles at z given by $z^2 - 1 = 0$

$\Rightarrow \quad z = \pm 1$

The circle with centre $z = 1$ and radius 1 encloses a simple pole at $z = 1$.

By Cauchy Integral formula

$$\int_C \frac{z^2+1}{z^2-1} dz = \int \frac{\{(z^2+1)/(z+1)\}}{z-1} dz$$

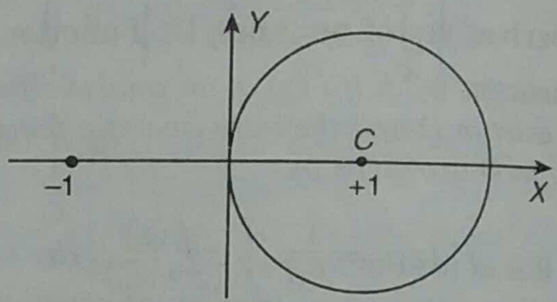


Fig. 6.13

Let $f(z) = \frac{z^2+1}{z+1}$

Then $\int_C \frac{z^2+1}{z^2-1} dz = \int_C \frac{f(z)}{z-1} dz$

$$= 2\pi i\left[\frac{z^2+1}{z+1}\right]_{z=1} = 2\pi i$$

Ex. 25. *Evaluate* $\int_C \frac{dz}{z^2-1}$, *where C is a circle* $x^2 + y^2 = 4$.

Solution. The poles of integrand are given by

$$z^2 - 1 = 0 \Rightarrow z = \pm 1$$

The given circle $x^2 + y^2 = 4$ with centre at $z = 0$ and radius 2, encloses both simple poles at $z = \pm 1$.

Eliminating these poles by drawing circles of very small radii and making cross cuts to form simply connected region, we get

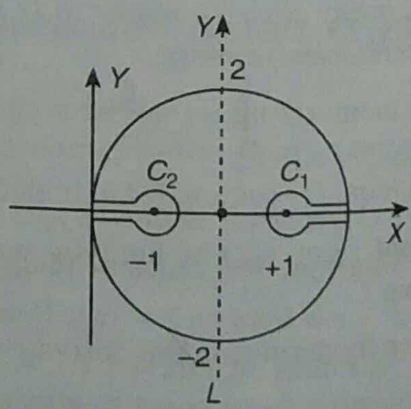


Fig. 6.14

$$\oint_C \frac{dz}{z^2-1} = \oint_{C_1} \frac{dz}{z^2-1} + \oint_{C_2} \frac{dz}{z^2-1}$$

$$= \oint_{C_1} \frac{dz/(z+1)}{z-1} + \oint_{C_2} \frac{dz/(z-1)}{z+1}$$

$$= 2\pi i\left\{\left[\frac{1}{z+1}\right]_{z=1} + \left[\frac{1}{z-1}\right]_{z=-1}\right\}$$

$$= 2\pi i\left[\frac{1}{2} - \frac{1}{2}\right] = \mathbf{0}$$

Ex. 26. *Evaluate the complex integral*

$$\int_C \tan z\, dz \text{ were } C \text{ is a circle } |z| = 2.$$

Solution. $\int_C \tan z\, dz = \int_C \frac{\sin z}{\cos z} dz$

The poles are given by $\cos z = 0$

$\Rightarrow \quad z = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

As radius of circle is $|z| = 2$, the only poles lying within the contour is $z = \frac{\pi}{2}$.

By Cauchy integral formula

$$\int_C \tan z\, dz = 2\pi i\, [\sin z]_{z=\frac{\pi}{2}} = 2\pi i \sin\frac{\pi}{2} = 2\pi i$$

6·17. Derivatives of an Analytic Function

Theorem 6·7 : *If $f(z)$ is an analytic function in a domain D, then its derivatives of all orders exist in D and they are analytic functions in D. The values of these derivatives at any point z_0 in D are given by*

$$f'(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2} dz \qquad \ldots (1a)$$

$$f''(z_0) = \frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^3} dz \qquad \ldots (1\,b)$$

and in general

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}} dz \ (n = 1, 2, 3, \ldots); \qquad \ldots (1\,c)$$

where C is any closed contour in D surrounding the point $z = z_0$; the contour C being traversed in counterclockwise sense. (*Rohilkhand 2008*)

For memorizing it may be noted that the values of the derivatives of analytic function $f(z)$ at any point z_0 in D are obtained formally by repeated differentiation of Cauchy's integral formula (eqn. (1) section 6·16, under the integral sign with respect to z_0).

Proof : Let $f(z)$ be analytic within and on a closed contour C and that the point $z_0 + \Delta z$ lie within C.

Then by definition the derivative of $f(z)$ at point z_0 is given by

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \qquad \ldots (2)$$

From Cauchy's integral formula, we have

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0} dz$$

and $$f(z_0 + \Delta z) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - (z_0 + \Delta z)} dz$$

$$\therefore \quad f'(z_0) = \lim_{\Delta z \to 0} \frac{1}{2\pi i \Delta z} \int_C \left(\frac{1}{z - z_0 - \Delta z} - \frac{1}{z - z_0} \right) f(z) dz$$

$$= \lim_{\Delta z \to 0} \frac{1}{2\pi i} \int_C \frac{f(z)\, dz}{(z - z_0 - \Delta z)(z - z_0)}$$

But straight-forward calculation shows that

$$\frac{1}{(z - z_0 - \Delta z)(z - z_0)} = \frac{1}{(z - z_0)^2} + \frac{\Delta z}{(z - z_0 - \Delta z)(z - z_0)^2}$$

$$\therefore \quad f'(z_0) = \lim_{\Delta z \to 0} \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^2} dz + \lim_{\Delta z \to 0} \frac{\Delta z}{2\pi i} \int_C \frac{f(z)\, dz}{(z - z_0 - \Delta z)(z - z_0)^2} \quad \ldots (3)$$

Since $f(z)$ is analytic within and on the closed contour C, we can find a positive number M such that $|f(z)| < M$ and let L be the length of C. Then if d is the shortest distance from z_0 to C and if $|\Delta z| < d$, we write

$$\left| \frac{\Delta z}{2\pi i} \int_C \frac{f(z)\, dz}{(z - z_0 - \Delta z)(z - z_0)^2} \right| < \frac{ML\, |\Delta z|}{2\pi\, d^2 (d - |\Delta z|)}$$

Now it is evident that the right hand side approaches zero when Δz approaches zero, consequently eqn. (3) gives

$$f'(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^2} dz \quad \ldots (1a)$$

Let us now proceed to prove formula (1b).

By definition

$$f''(z_0) = \lim_{\Delta z \to 0} \frac{f'(z_0 + \Delta z) - f'(z_0)}{\Delta z}$$

From (1a)

$$f'(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^2} dz$$

$$f'(z_0 + \Delta z) = \frac{1}{2\pi i} \int_C \frac{f(z)\, dz}{[z - (z_0 + \Delta z)]^2}$$

$$\therefore \quad f''(z_0) = \lim_{\Delta z \to 0} \frac{f'(z_0 + \Delta z) - f'(z_0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{1}{2\pi i\, \Delta z} \int_C \left[\frac{1}{(z - z_0 - \Delta z)^2} - \frac{1}{(z - z_0)^2} \right] f(z)\, dz \quad \ldots (4)$$

Straight-forward calculation shows that,

$$\frac{1}{\Delta z}\left[\frac{1}{(z-z_0-\Delta z)^2}-\frac{1}{(z-z_0)^2}\right]=\frac{2}{(z-z_0)^3}+\frac{\Delta z\{3(z-z_0)-2\Delta z\}}{(z-z_0-\Delta z)(z-z_0)^2}$$

Hence equation (3) may be written in the form

$$f''(z_0)=\lim_{\Delta z\to 0}\frac{2!}{2\pi i}\int_C\frac{f(z)\,dz}{(z-z_0)^3}+\lim_{\Delta z\to 0}\frac{\Delta z}{2\pi i}\int_C\frac{\{3(z-z_0)-2\Delta z\}\,f(z)}{(z-z_0-\Delta z)\,(z-z_0)^3}\,dz \quad \dots (5)$$

Since $f(z)$ is analytic, we can find a positive number M' such that $|\{3(z-z_0)-2\Delta z\}\,f(z)| < M'$; hence

$$\left|\frac{\Delta z}{2\pi i}\int_C\frac{\{3(z-z_0)-2\Delta z\}\,f(z)}{(z-z_0-\Delta z)\,(z-z_0)^3}\,dz\right|<\left|\frac{M'L\,|\Delta z|}{2\pi d^3\,(d-|\Delta z|)}\right|$$

Evidently, right hand side approaches zero when Δz approaches zero, consequently eqn. (5) gives

$$f''(z_0)=\frac{2!}{2\pi i}\int_C\frac{f(z)}{(z-z_0)^3}\,dz \quad \dots (1b)$$

Thus, there exists the derivatives of $f'(z)$ at each point z_0 interior to the region bounded by the closed contour C ; thereby indicating that $f'(z)$ is analytic at each point within and on the closed contour C.

The argument used in establishing formulae (1*a*) and (1 *b*) can be applied successively to obtain a formula for the derivative of any given order. By mathematical induction we obtain the general formula (1*c*) *i.e.*,

$$f^{(n)}(z_0)=\frac{n!}{2\pi i}\int_C\frac{f(z)\,dz}{(z-z_0)^{n+1}}\ (n=1,2,3,\ldots.) \quad \dots (1c)$$

That is, if we assume that this formula is true for any particular integer $m=k$ then by proceeding as before we can show that it is true for $n=k+1$. *Thus the requirement that $f(z)$ be analytic not only guarantees a first derivative but derivatives of all orders as well, which imply that the derivatives of $f(z)$ are automatically analytic.* This statement assumes the Goursat version of the Cauchy's integral theorem.

Ex.27. *By using the integral representation of $f^n(a)$ prove that*

$$\sum_{n=0}^{\infty}\left(\frac{x^n}{n!}\right)=\frac{1}{2\pi}\int_0^{2\pi}e^{2x\cos\theta}\,d\theta$$ (*Rohilkhand 1998*)

Solution. The integral representation of $f^n(a)$ is

$$f^n(a)=\frac{n!}{2\pi i}\int_C\frac{f(z)\,dz}{(z-a)^{n+1}} \quad \dots (1)$$

where C is any closed contour surrounding the origin

$$\therefore\quad f^n(0)=\frac{n!}{2\pi i}\int_C\frac{f(z)\,dz}{z^{n+1}} \quad \dots (2)$$

Let us assume $f(z)=e^{xz}$

$$= \frac{2\pi i}{2\,!} \lim_{z \to 1} \frac{d^2}{dz^2} (5z^2 - 3z + 2) = \frac{2\pi i}{2\,!} \lim_{z \to 1} (10z - 3)$$

$$= \frac{2\pi i}{2\,!} \lim_{z \to 1} (10) = \mathbf{10}\ \pi i$$

6·18. Morera's Theorem

(Converse of Cauchy's Theorem)

Theorem 6·9. *If a function $f(z)$ is continuous in a simply connected domain D and if $\oint_C f(z)\,dz = 0$ for every closed contour C in domain D, then $f(z)$ is analytic throughout D.*

This theorem is the converse of Cauchy's integral theorem.

Proof. In section 6·15 we have proved that the derivative of the function

$$F(z) = \int_{z_0}^{z} f(\xi)\, d\xi$$

exist at each point in a simply connected domain D, in fact, that

$$F'(z) = f(z)$$

In our proof we assumed there that $f(z)$ is analytic in domain D ; but we used only two properties of $f(z)$, namely that it is continuous in D and that its integral around closed contour D vanishes. In theorem 6·9 these two properties of $f(z)$ are assumed, hence we conclude that *the derivative of $f(z)$ exists at each point in D and is given by $F'(z) = f(z)$* at each point z in D.

This implies that $F(z)$ is analytic in D. Further, since (from section 6·17) *the derivative of every analytic function is itself analytic* and here $f(z)$ appears as a derivative of analytic function $F(z)$; hence *it follows that the function $f(z)$ is analytic in D*. This proves Morera's theorem.

6·19. Liouville's Theorem

Theorem 6·10. *If $f(z)$ is analytic and bounded in absolute value for all (finite) values of z in the complex plane, then $f(z)$ is a constant.*

Proof. Let C be a circle of radius r with centre at z_0 *i.e.*, $|z - z_0| = r$ on C. By assumption $|f(z)|$ is bounded for all values of z, therefore, a constant M exists such that $|f(z)| \leq M$ for all z within and on C.

As $f(z)$ is analytic within and on C, therefore, according to integral formula for derivatives

$$f^n(z) = \frac{n\,!}{2\pi i} \int_C \frac{f(z)\,dz}{(z - z_0)^{n+1}} \quad (n = 1, 2, 3..)$$

Then we get

$$\left| f^n(z_0) \right| = \frac{n\,!}{2\pi} \left| \int_C \frac{f(z)\,dz}{(z - z_0)^{n+1}} \right| \leq \frac{n\,!}{2\pi} \frac{M \cdot 2\pi r}{r^{n+1}}$$

This yields **Cauchy's inequality**

$$\left| f^n(z_0) \right| \leq \frac{n!\,M}{r^n} \qquad \text{...(1)}$$

For $n = 1$, we see that at each point z_0 and for every positive r

$$f'(z_0) \leq \frac{M}{r} \qquad \text{...(2)}$$

We can take r as large as we please. Since $f'(z)$ is a fixed number, we conclude that $f'(z) = 0$ and since z_0 is arbitrary, therefore, $f'(z) = 0$ for all finite z and hence $f(z)$ is constant. This proves Liouville's theorem.

6·20 Taylor's Series

Theorem 6·11. *If $f(z)$ is analytic at all points inside a circular domain D with its centre at $z = z_0$ and radius r_0, then for every z inside D,*

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \ldots + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n \qquad \text{...(1)}$$

This series is called the **Taylor's series** *of $f(z)$ with centre at z_0.*

In particular case when $z_0 = 0$, this series is called the **Maclaurin series** *of $f(z)$.*

Proof. Let C_0 denote the boundary of the circular domain D of radius r_0. Let z be any fixed point inside the circle C_0 and such that $|z - z_0| = r$ where $r < r_0$. Let z' denote any point on a circle $|z' - z_0| = r_1$, denoted by C where $r < r_1 < r_0$. Then, as shown in Fig. 6·15, z is inside C and $f(z)$ is analytic within and on C; hence from Cauchy's integral formula it follows that

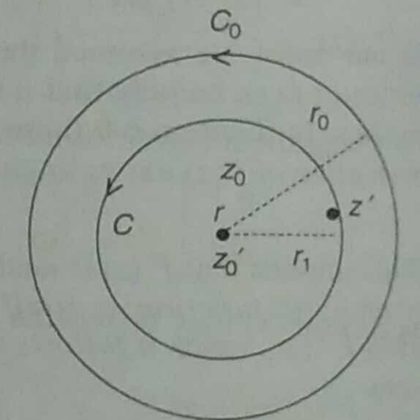


Fig. 6·15

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(z')\,dz'}{z' - z}$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z')\,dz'}{(z' - z_0) - (z - z_0)}$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z')\,dz'}{(z' - z_0)\left[1 - \dfrac{z - z_0}{z' - z_0}\right]} \qquad \text{...(2)}$$

From geometric progression

$$1 + q + q^2 + \ldots + q^n = \frac{1 - q^{n+1}}{1 - q}$$

We obtain the relation

$$\frac{1}{1 - q} = 1 + q + q^2 + \ldots + q^n + \frac{q^{n+1}}{1 - q} \qquad \text{...(3)}$$

Now for point z interior to C, we have $|z - z_0| < |z' - z_0|$ since z' is on C.

i.e., $\left|\dfrac{z - z_0}{z' - z_0}\right| < 1$

By setting $q = \dfrac{z - z_0}{z' - z_0}$ in (3), we get

$$\frac{1}{1 - \left[\dfrac{(z - z_0)}{(z' - z_0)}\right]} = 1 + \frac{z - z_0}{z' - z_0} + \left(\frac{z - z_0}{z' - z_0}\right)^2 + \ldots + \left(\frac{z - z_0}{z' - z_0}\right)^n$$

$$+ \frac{(z - z_0) / (z' - z_0)^{n+1}}{\left[1 - \dfrac{z - z_0}{z' - z_0}\right]} \quad \ldots (4)$$

Substituting this in equation (2) and taking powers of $(z - z_0)$ out from under the integral sign, since z and z_0 are constants, we get

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(z')\, dz'}{z' - z_0} + \frac{(z - z_0)}{2\pi i} \int_C \frac{f(z')\, dz'}{(z' - z_0)^2} + \cdots$$

$$+ \frac{(z - z_0)^n}{2\pi i} \int_C \frac{f(z')}{(z' - z_0)^{n+1}}\, dz' + R_n(z) \quad \ldots (5)$$

where $R_n(z)$ is given by

$$R_n(z) = \frac{(z - z_0)^{n+1}}{2\pi i} \int_C \frac{f(z')\, dz'}{(z' - z_0)^{n+1} (z' - z)} \quad \ldots (6)$$

Using the integral formulae (1) of section 6·17, *viz.*,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)\, dz}{(z - z_0)^{n+1}} \; (n = 1, 2, 3 \ldots) \quad \ldots (7)$$

expression (5) may be written in the form

$$f(z) = f(z_0) + \frac{(z - z_0)}{1!} f'(z_0) + \frac{(z - z_0)^2}{2!} f''(z_0) + \ldots + \frac{(z - z_0)^n}{n!} f^{(n)} z_0 + R_n \quad \ldots (8)$$

This expression is called *Taylor's formula with remainder* R_n.

Since $f(z)$ is analytic, it has derivatives of all orders ; hence we may take n in eqn. (8) as large as we please. If we let n approach infinity, we obtain from (8) the power series

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n \quad \ldots (9)$$

Clearly this series will converge and represent $f(z)$ if and only if

$$\lim_{n \to \infty} R_n(z) = 0 \quad \ldots (10)$$

Since $|z - z_0| = r$, $|z' - z_0| = r_1$ and $|z' - z| \geq r_1 - r$, it follows from equation (6) that when M denotes the maximum value of $f(z')$ on C,

$$|R_n| \leq \frac{r^{n+1}}{2\pi} \cdot \frac{M \cdot 2\pi r_1}{r_1^{n+1} (r_1 - r)} = \frac{r_1 M}{(r_1 - r)} \cdot \left(\frac{r}{r_1}\right)^{n+1}$$

But $\dfrac{r}{r_1} < 1$, therefore if we let n approach infinity, the expression on right approaches zero. This proves (10) for all values of z inside C. It follows that series (9) converges and represents $f(z)$. It may be noted that the expansion (9) of $f(z)$ is based on the assumption that $f(z)$ is analytic for $|z - z_0| < r_0$. *The representation of $f(z)$ in the form* (9) *is unique* in the sense that (9) is the only power series with centre at z_0 which represents the given function $f(z)$ *and is the desired Taylor's series of $f(z)$ with centre at z_0*. This proves theorem 6·11.

When $z_0 = 0$; the series (9) reduces to *Maclaurin series*

$$f(z) = f(0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n\,!} z^n \qquad \text{... (11)}$$

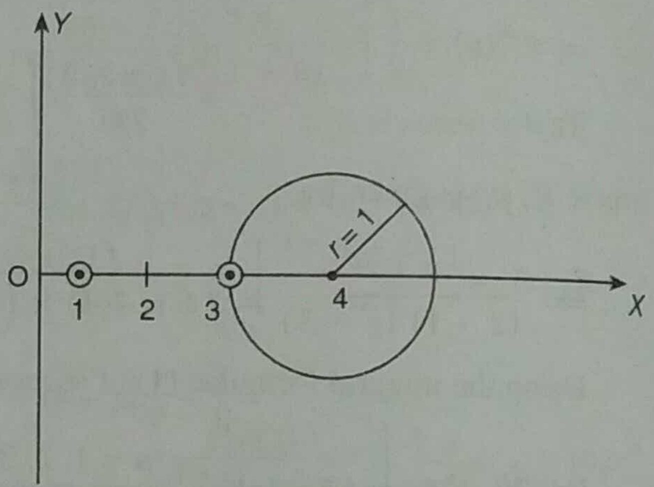
Ex. 29. *Find the Taylor series expansion of a function of the complex variable*

$$f(z) = \frac{1}{(z-1)(z-3)}$$

about the point $z = 4$. Find its region of convergence. (*Awadh 2008*)

Solution. $f(z) = \dfrac{1}{(z-1)(z-3)}$

the singular points are $z = 1$ and $z = 3$. If the centre of the circle is at $z = 4$, then its distances of singular points from centre are 3 and 1. Hence if a circle is drawn with centre at $z = 4$, then within the circle $|z - 4| = 1$, the given function is analytic, hence it can be expanded in taylor series within the circle $|z - 4| = 1$; which is the *circle of convergence*



$$\therefore \quad f(z) = \frac{1}{(z-1)(z-3)}$$

$$= \frac{1}{2}\left(\frac{1}{z-3} - \frac{1}{z-1}\right)$$

$$= \frac{1}{2}\left\{\frac{1}{(2-4)+1} - \frac{1}{(2-4)+3}\right\}$$

$$= \frac{1}{2}\{1 + (2-4)\}^{-1} - \frac{1}{6}\left\{1 + \frac{(2-4)}{3}\right\}^{-1}$$

$$= \frac{1}{2}[1 - (z-4) + (z-4)^2 - (z-4)^3 + -]$$

$$- \frac{1}{6}\left[1 - \frac{z-4}{3} + \frac{(z-4)^3}{9} - (z-4)^3 + \ldots\right]$$

$$= \left(\frac{1}{2} - \frac{1}{6}\right) + \left(-\frac{1}{2} + \frac{1}{18}\right)(z-4) + \left(\frac{1}{2} - \frac{1}{54}\right)(z-4)^2 + \left(-\frac{1}{2} + \frac{1}{162}\right)(z-4)^3$$

$+$

$$f(z) = \frac{1}{3} - \frac{4}{9}(z-4) + \frac{13}{27}(z-4)^2 - \frac{40}{81}(z-4)^3 +$$

Aliter

$$f(z) = \frac{1}{(z-1)(z-3)},\; f(u) = \frac{1}{(u-1)(u-3)} = \frac{1}{3}$$

The given function may be expressed in the form of partial fractions as

$$f(z) = \frac{1}{2}\left(\frac{1}{z-3} - \frac{1}{z-1}\right)$$

$$f'(z) = \left(\frac{\partial f}{\partial z}\right) = \frac{1}{2}\left[-\frac{1}{(2-3)^2} + \frac{1}{(z-1)^2}\right]$$

$$\Rightarrow f'(u) = \frac{1}{2}\left[-\frac{1}{(u-3)^2} + \frac{1}{(u-1)^2}\right] = -\frac{4}{9}$$

$$f''(z) = \left(\frac{\partial^2 f}{\partial z^2}\right) = \frac{1}{2}\left[\frac{2}{(z-3)^3} - \frac{2}{(z-1)^3}\right]$$

$$\Rightarrow f''(u) = \frac{1}{2}\left[\frac{2}{(u-3)^3} - \frac{2}{(h-1)^3}\right] = \frac{26}{27}$$

$$f'''(z) = \frac{1}{2}\left[-\frac{6}{(z-1)^4} + \frac{6}{(z-1)^4}\right]$$

$$\Rightarrow f'''(u) = \frac{1}{2}\left[-\frac{6}{(u-1)^4} + \frac{6}{(u-1)^4}\right] = -\frac{80}{27}$$

Taylor series is

$$f(z) = f(z_0) + (z - z_0) f'(z_0) + \frac{(z-z_0)^2}{2} f''(z_0) + \frac{(z-z_0)^3}{3!} f'''(z_0) + \ldots$$

$$\Rightarrow \quad \frac{1}{(z-1)(z-3)} = \frac{1}{3} + (z-4) \times \left(-\frac{4}{9}\right) + \frac{(z-4)^2}{2!}\left(\frac{26}{27}\right) + \frac{(z-4)^3}{3!}\left(-\frac{80}{27}\right)$$

$$= \frac{1}{3} - \frac{4}{9}(z-4) + \frac{13}{27}(z-4)^2 - \frac{40}{81}(z-4)^3$$

Ex. 30. *Find the first three terms of the Taylor series expansion of* $f(z) = \dfrac{1}{z^2+4}$ *about* $z = -i$. *Also find the region of convergence.* (*Bombay 2005*)

Solution. $f(z) = \dfrac{1}{z^2+4}$

Poles are given by $z^2 + 4 = 0 \Rightarrow z^2 = -4$ or $z = \pm 2i$. If the centre of circle is about $z = -i$, then the distance of the singularities $z = 2i$ and $z = -2i$ from centre are 3 and 1. Hence if we draw a circle of radius 1, then within the circle $|z + i| = 1$, the given function is analytic. Hence the function can be expanded as a Taylor series within the circle $|z + i| = 1$; which is the region of convergence.

By Taylor's series,

$$f(z) = f(z_0) + (z - z_0) f'(z_0) + \frac{(z-z_0)^2}{2!} f''(z_0) + \frac{(z-z_0)^3}{3!} f'''(z_0)$$

$$\therefore \quad f(z) = f(-i) + (z+i) f'(-i) + \frac{(z+i)^2}{2!} f''(-i) + \ldots \quad (1)$$

$$f(z) = \frac{1}{z^2+4} \Rightarrow f(-i) = \frac{1}{(-1)+4} = \frac{1}{3}$$

$$f'(z) = -\frac{2z}{(z^2+4)^2} = -\frac{2 \times -i}{(-1+4)^2} = \frac{2i}{9}$$

$$f''(z) = -\frac{(z^2+4)^2(2) - 2z \cdot 2(z^2+4) \cdot 2z}{(z^2+4)^4} = -\frac{2(z^2+4) - 8z^2}{(z^2+4)^3}$$

$$\Rightarrow \quad f''(-i) = -\frac{2(-1+4) - 8(-1)}{(-1+4)^3} = -\frac{14}{27}$$

Substituting, these values in first three terms of Taylor's series (1), we obtain

$$f(z) = \frac{1}{3} + (z+i)\left(\frac{2i}{9}\right) - \frac{14}{27}(z+i)^2 + \ldots$$

Ex. 31. *Expand $f(z) = \sin z$ in Taylor series about (i) $z = 0$ and (ii) $z = \pi/4$.* (Avadh 2010)

Solution. Taylor series of the function $f(z)$ about $z = a$ is

$$f(z) = f(a) + f'(a)(z-a) + f''(a)\frac{(z-a)^2}{21} + \ldots + \frac{f^n}{n!}(z-a)^n + \ldots$$

Given $f(z) = \sin z$

$f'(z) = \cos z, f''(z) = -\sin z, f'''(z) = -\cos z$

$f^{(iv)}(z) = -\sin z,\ f^{(v)}(z) = \cos z$ etc.

(1) For $z = 0$,

$f(0) = 0, f'(0) = 1, f''(0) = 0, f'''(0) = -1$

$f^{(iv)}(0) = 0,\ f^{(V)} = 1, f^{(2n-1)}(0) = (-1)^{n+1}$

As $f(z)$ is analytic for all values of z, therefore applying Taylor's Series, we get

$$f(z) = \sin z = f(0) + z(f'(0) + \frac{z^2}{21}f''(0) + \frac{z^3}{3!}f'''(0) + \ldots + \frac{z^{2n-1}}{(2n-1)!}f^{(2nd)}(0)$$

$$= 0 + z + 0 - \frac{z^3}{3!} + 0 + \frac{z^5}{5!} + \ldots + \frac{z^{2n-1}}{(2n-1)!}(-1)^{n+1}\ldots$$

$$= \sum_{n=1}^{n} (-1)^{n+1}\frac{z^{2n-1}}{(2n-1)!} \text{ where } |z| < \infty$$

(*ii*) For $z = \pi/4$

$$f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}},\ f'\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$f''\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}},\ f'''\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}},\ f^{(iv)}\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \text{ etc.}$$

$$f(z) = \sin z = \frac{1}{\sqrt{2}} + \left(z - \frac{\pi}{4}\right) - \frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}\,2!}\left(z - \frac{\pi}{4}\right)^2 + \frac{1}{\sqrt{2}\,3!}\left(z - \frac{\pi}{4}\right)^3 + \ldots\left(z - \frac{\pi}{4}\right)^{2n-1}$$

6·21. Laurent's series

In many cases it is necessary to expand a function $f(z)$ around points where $f(z)$ is singular, so that Taylor's theorem is inapplicable. For example, if $f(z)$ is analytic in an annulus bounded by two concentric circles C_1 and C_2 and at each point on C_1 and C_2, then Taylor's theorem can not be applied and hence a new type of series representation of $f(z)$, given by Laurent, is necessary.

Theorem. 6·12. *If $f(z)$ is analytic and single-valued on two concentric circles C_1 and C_2 with centre at z_0 and in the annulus between them then $f(z)$ can be represented by Laurent's series*

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n} \qquad \ldots (1)$$

where $$a_n = \frac{1}{2\pi i}\int_C \frac{f(z')\,dz'}{(z'-z_0)^{n+1}}, \qquad b_n = \frac{1}{2\pi i}\int_C (z'-z_0)^{n-1} f(z')\,dz' \qquad \dots (2)$$

Instead of (1) Laurent's series may be expressed as

$$f(z) = \sum_{n=-\infty}^{+\infty} A_n (z-z_0)^n \qquad \dots (3)$$

where $$A_n = \frac{1}{2\pi i}\int_C \frac{f(z')\,dz'}{(z'-z_0)^{n+1}} \qquad \dots (4)$$

Proof. Let $f(z)$ be analytic in the annular region between and on two concentric circles C_1 and C_2 of radii r_1 and r_2 respectively with centre at z_0. Let us draw an imaginary cut line to convert given region into a simply connected one ; so that Cauchy's integral formula is applicable. From which, at each point z in the given annulus it follows that

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(z')\,dz'}{z'-z} - \frac{1}{2\pi i}\int_{C_2} \frac{f(z')\,dz'}{z'-z} \qquad \dots (5)$$

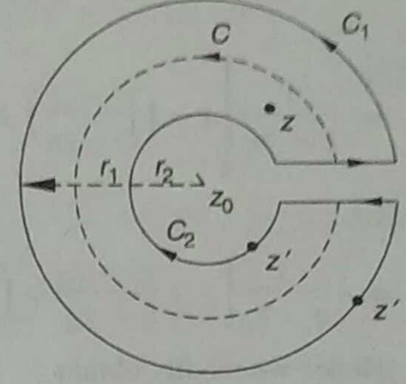


Fig. 6·16

Here negative sign is introduced because contours C_1 and C_2 are to be traversed in counterclockwise sense. As z lies inside C_1, the first integral in equation (5) may be developed like that of Taylor's series Therefore we write

$$\frac{1}{z'-z} = \frac{1}{(z'-z)-(z-z_0)} = \frac{1}{(z'-z_0)\left[1-\left(\frac{z-z_0}{z'-z_0}\right)\right]}$$

$$= \frac{1}{(z'-z_0)}\left[1+\left(\frac{z-z_0}{z'-z_0}\right)+\left(\frac{z-z_0}{z'-z_0}\right)^2+\dots+\left(\frac{z-z_0}{z'-z_0}\right)^n+\frac{(z-z_0)^{n+1}}{(z'-z_0)^n(z'-z)}\right]$$

Therefore

$$\frac{1}{2\pi i}\int_{C_1}\frac{f(z')\,dz'}{z'-z} = \frac{1}{2\pi i}\int_{C_1}\frac{f(z')\,dz'}{(z'-z_0)} + \frac{(z-z_0)}{2\pi i}\int_{C_1}\frac{f(z')\,dz'}{(z'-z_0)^2}+\dots$$

$$+\frac{(z-z_0)^n}{2\pi i}\int_{C_1}\frac{f(z')\,dz'}{(z'-z_0)^{n+1}} + R_n(z)$$

where the remainder $R_n(z)$ is given by

$$R_n(z) = \frac{(z-z_0)^{n+1}}{2\pi i}\int_{C_1}\frac{f(z')\,dz'}{(z'-z_0)^{n+1}(z'-z)}$$

Now estimating the remainder $R_n(z)$ as in Taylor's series we get $R_n(z) \to 0$ as $n \to \infty$. Then we obtain

$$\frac{1}{2\pi i}\int_{C_1}\frac{f(z')\,dz'}{z'-z} = \sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad \dots (6)$$

where the coefficient a_n are given by

$$a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(z')\, dz'}{(z' - z_0)^{n+1}} \quad \text{... (7)}$$

In the second integral of equation (5) ; the situation is different, since z lies outside C_2 ; therefore in this case $\left| \frac{z' - z_0}{z - z_0} \right| < 1$ *i.e.*, we shall now develop $\frac{1}{z' - z}$ in powers of $\frac{z' - z_0}{z - z_0}$ for the resulting series to be convergent. Therefore in the second integral we write

$$\frac{1}{z' - z} = -\frac{1}{z - z'} = -\frac{1}{(z - z_0) - (z' - z_0)} = \frac{1}{(z - z_0)\left[1 - \left(\frac{z' - z_0}{z - z_0}\right)\right]}$$

$$= -\frac{1}{z - z_0}\left[1 + \left(\frac{z' - z}{z - z_0}\right) + \left(\frac{z' - z_0}{z - z_0}\right)^2 + \ldots + \left(\frac{z' - z_0}{z - z_0}\right)^n + \frac{[(z' - z_0)/(z - z_0)]^{n+1}}{1 - \left(\frac{z' - z_0}{z - z_0}\right)}\right]$$

$$= -\frac{1}{z - z_0}\left[1 + \frac{z' - z_0}{z - z_0} + \left(\frac{z' - z_0}{z - z_0}\right)^2 + \ldots + \left(\frac{z' - z_0}{z - z_0}\right)^n\right] - \frac{1}{(z - z')}\left(\frac{z' - z_0}{z - z_0}\right)^{n+1}$$

Hence we readily obtain

$$-\frac{1}{2\pi i}\int_{C_2} \frac{f(z')\, dz'}{z' - z} = \frac{1}{2\pi i}\left[\frac{1}{(z - z_0)}\int_{C_2} f(z')\, dz' + \frac{1}{(z - z_0)^2}\int_{C_2} (z' - z_0) f(z')\, dz' + \ldots + \right.$$

$$\left. + \frac{1}{(z - z_0)^n}\int_{C_2} (z' - z_0)^{n-1} f(z')\, dz' + \frac{1}{(z - z_0)^{n+1}}\int_{C_2} (z' - z_0)^n f(z')\, dz'\right] + Q_n(z) \quad \text{... (8)}$$

where the remainder $Q_n(z)$ is given by

$$Q_n(z) = \frac{1}{2\pi i\,(z - z_0)^{n+1}}\int \frac{(z' - z_0)^{n+1}}{(z - z')} f(z')\, dz' \quad \text{... (9)}$$

To estimate the remainder let $r = |z - z_0|$; then $r_2 < r < r_1$ where $r_2 = |z' - z_0|$; and $|z - z'| \geq r - r_2$. If M is the maximum value of $f(z')$ on C_2 ; then it follows from equation (9) that

$$|Q_n(z)| \leq \frac{1}{2\pi r^{n+1}} \frac{M \cdot 2\pi r_2\, r_2^{n+1}}{r - r_2} = \left(\frac{r_2}{r}\right)^{n+1} \frac{M r_2}{r - r_2}$$

As $\frac{r_2}{r} < 1$; therefore $Q_n(z)$ approaches zero as n tends to infinity. In view of this, equation (8) can be written in the form

$$-\frac{1}{2\pi i}\int_{C_2} \frac{f(z')\, dz'}{z' - z} = \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \quad \text{... (10)}$$

where the coefficients b_n are given by $b_n = \frac{1}{2\pi i}\int_{C_2} (z' - z_0)^{n-1} f(z')\, dz'$ ... (11)

Since z_0 is not a point of the annulus, the functions $\frac{f(z')}{(z'-z_0)^{n+1}}$ and $(z'-z_0)^{n-1} f(z')$ for all values of n are analytic in the annulus, hence we see that for every contour C within the given annulus with its centre at z_0

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z')\,dz'}{(z'-z_0)^n} \text{ for all } n \geq 0$$

$$b_n = \frac{1}{2\pi i}\int_C (z'-z_0)^{n+1} f(z')\,dz' \text{ for all } n \geq 1$$

Therefore from equations (5), (6) and (10), we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n} \quad \text{... (12)}$$

where a_n and b_n are given by equations (7) and (11).

Equation (12) represents *the Laurent's series of given analytic function f (z) in its annulus of convergence.*

Instead of (12) the Laurent's series can be put in a uniform form as follows :

$$f(z) = \sum_{n=-\infty}^{+\infty} A_n (z-z_0)^n \quad \text{... (13)}$$

where the coefficients A_n are given by

$$A_n = \frac{1}{2\pi i}\int_C \frac{f(z')\,dz}{(z'-z_0)^{n+1}} \quad \text{... (14)}$$

where n is any integer positive, negative or zero.

It may be noted that the Laurent's series of a given function $f(z)$ in its annulus of convergence is unique. However, $f(z)$ may have different Laurent's series in two annuli with the same centre. If a Laurent's series is found by any method, then according to uniqueness property, it must be then Laurent series of given function in given annulus.

Ex. 32. *Find all Laurent series of function* $f(z) = \frac{1}{(1-z^2)}$ *with centre at* $z = 1$.

Solution. We have

$$f(z) = \frac{1}{1-z^2} = -\frac{1}{1-z^2} = -\frac{1}{(z-1)(z+1)}$$

(*i*) Using the geometric series

$$\frac{1}{1-q} = \sum_{n=0}^{\infty} q^n$$

we get

$$\frac{1}{z+1} = \frac{1}{2+(z-1)} = \frac{1}{2}\frac{1}{\left[1-\left\{-\frac{(z-1)}{2}\right\}\right]}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left\{-\frac{(z-1)}{2}\right\}^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(z-1)^n \quad \text{... (1)}$$

$$\Rightarrow \quad \frac{dz}{z} = i\, d\theta$$

we get

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} \frac{\cosh\,(e^{i\theta}+e^{-i\theta})e^{i\theta}\,d\theta}{e^{i(n+1)\theta}} = \frac{1}{2\pi}\int_0^{2\pi} \cosh\,(2\cos\theta)\, e^{-ni\theta}\, d\theta \qquad \text{... (6)}$$

and

$$b_n = \frac{1}{2\pi}\int_0^{2\pi} \cosh\,(e^{i\theta}+e^{-i\theta})\, e^{ni\theta}\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} \cosh\,(2\cos\theta)\, e^{in\theta}\, d\theta \qquad \text{... (7)}$$

As $a_n = b_n$, adding (6) and (7), we get

$$2\,a_n = 2b_n = \frac{1}{2\pi}\int_0^{2\pi} \cosh\,(2\cos\theta)\,(e^{in\theta}+e^{-in\theta})\, d\theta$$

$$\Rightarrow a_n = b_n = \frac{1}{2\pi}\int_0^{2\pi} \cosh\,(2\cos\theta)\cos n\theta\, d\theta \qquad \text{(Hence proved)}$$

6·22. Singularities of an Analytic Function

A singularity (or a singular point) of an analytic function f (z) is a point where f (z) ceases to be analytic. We can also say that $f(z)$ is singular or has a singularity at that point. The function $f(z)$ is said to be singular at infinity if $f\left(\frac{1}{z}\right)$ is singular at $z = 0$. In a given z-plane an analytic function may have a number of singularities.

Isolated Singularities. The point z_0 is defined as an isolated singularity of the function $f(z)$ if $f(z)$ is not analytic in the neighbourhood of $z = z_0$.

Types of Singularities. *Analytic functions may have two types of singularities* if we restrict ourselves only to single-valued functions :

1. Poles or non-essential singularities,
2. Essential singularities.

The distinction between these two types of singularities may be explained as follows :

Let $f(z)$ have an isolated singularity at $z = z_0$, then it can be represented by its *Laurent series* about z_0,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \qquad \text{... (1)}$$

which is valid throughout some neighbourhood of $z = z_0$ (except at $z = z_0$ itself). The last series in (1) is called the *principal part* of $f(z)$ near $z = z_0$.

Now there are *three* possibilities :

1. *The principal part of f (z) consists of only finite number of terms.* The equation (1) reduces to the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \ldots + \frac{b_m}{(z-z_0)^m}$$

where $b_m \neq 0$ and $b_n = 0$ for all $n > m$.

In this case, where the principal part consists of only a finite number of terms, the singularity of the function $f(z)$ at $z = z_0$ is called a pole and m is called the *order of the pole* and the function $f(z)$ is said to have a pole of mth order at the point z_0. Poles of first order (*i.e.*, $m = 1$) are called *simple poles.*

2. *The principal part of $f(z)$ consists of infinite number of terms.* Then Laurent expansion of $f(z)$ about z_0 is of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} = \sum_{n=-\infty}^{+\infty} A_n (z - z_0)^n \quad \text{... (2)}$$

In this case the singularity of the function $f(z)$ at $z = z_0$ is called the *isolated essential singularity.*

Any singularity of a single-valued analytic function other than a pole is called an *essential singularity.* Poles by definition, are isolated singularities. All singularities which are not isolated are thus essential singularities. An essential singularity may be isolated or not.

3. *The principal part contains no term i.e.,* all the coefficients b_n are zero. Then the Laurent expansion of $f(z)$ about z_0 is of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

Obviously, the function $f(z)$ is not analytic at $z = z_0$, but can be made analytic thereby assigning some value to $f(z)$ at $z = z_0$. Thus the function $f(z)$ has a singularity at $z = z_0$ which can be made to disappear by defining the function suitably. In this case the singularity of the function $f(z)$ at $z = z_0$ is called the *removable singularity*. Such singularities are of no interest, because they can be removed.

Entire Function. A function which is analytic everywhere in the finite plane is called an *entire function.* If such a function is also analytic at infinity, it is bounded for all z and from Liouville's theorem it follows that it must be a constant. Hence any entire function which is not a constant must be singular at infinity. For example, polynomials of at least the first degree e^z, $\sin z$ and $\cos z$ are entire functions and they are singular at infinity.

Meromorphic function. An analytic function whose only singularities in the finite plane are poles is said to be a meromorphic function.

Branch Point. This is a kind of singularity and is quite different from the poles. If we restrict to single valued functions, the singularity is the pole.

A multi-valued function $f(z)$ usually possesses two or more distinct values for each value of z. The important examples of multivalued functions are $\sqrt{z}$, $\sqrt[n]{z}$, $\sqrt{(z-a)(z-b)}$, $\log z$, z^{∞}, $\sin^{-1}(z)$, $\cos^{-1}(z)$. In order to apply the theorems of analytic function theory to such multivalued functions, we confine to their single valued branches where the function has continuous derivative and satisfying Cauchy Riemann equations and so the function is analytic in this region.

For example, $f(z) = \sqrt{z} = r^{1/2}\left[\cos\left(\frac{\theta + 2k\pi}{2}\right) + \sin\left(\frac{\theta + 2k\pi}{2}\right)\right.$, $k = 0, 1$ has two single valued branches $f_1(z) = r^{1/2} e^{i\theta/2}$ and $f_2(z) = r^{1/2} e^{i\left(\frac{\theta + 2\pi}{2}\right)}$, each has derivative $f'(z) = \frac{1}{2\sqrt{2}}$. Thus each of these branches is analytic except at $z = 0$. Another example

$f(z) = \log z = \log r + (\theta + 2k\pi) i$, $k = 0, \pm 1, \pm 2, \ldots$

has infinitely many branches, each of which has a derivative $f'(z) = \frac{1}{z}$ and at $z = 0$, the branches of log z are not analytic. Each branch of $\sqrt{z}$ and log z is discontinuous in the neighbourhood of its singular point. The points at which the values of branches of multiple valued functions become equal (or infinite) are called the branch points. The branch points are not the isolated singular points of $f(z)$. In general $f(z) = z^a$ where a is not integer, has a branch point.

For example $f(z) = (z^2 - 1)^{1/2} = (z - 1)^{1/2} (z + 1)^{1/2}$ has branch point singularities at $z = +1$ and $z = -1$.

6·23. Conformal Mapping

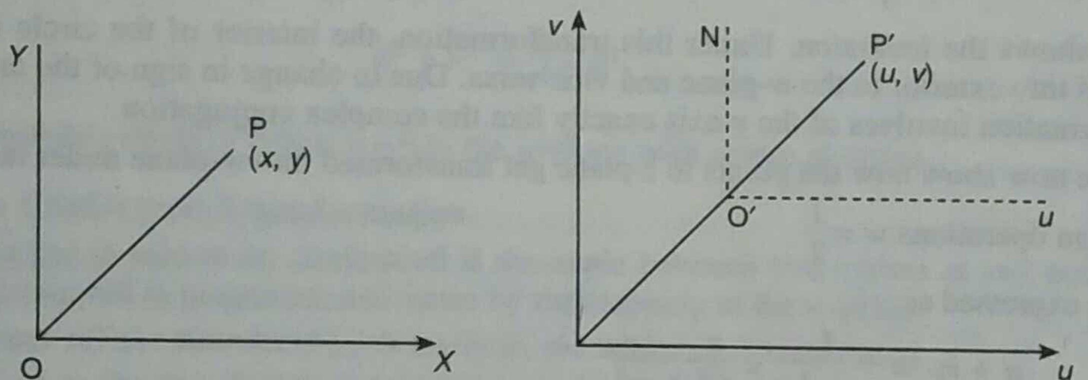
A complex number $z = x + iy$ may be plotted in the z-plane. The function $f(z) = W = u + iv$ may be plotted in a separate plane called the *W-plane. For a point in the* z-plane (specific values of *ax* and *y*), there may correspond the specific values of $u(x,y)$ and $v(x, y)$ which may yield a point in the *W*-plane,. Thus the points in the z-plane may be transformed or mapped into points in the *W*-plane, the correspondence between a point (or curve) in the z-plane and the image point (or curve) in the *W*-plane is called a **mapping.**

We consider the following operations for transformation from z-plane to *W-plane.*

(*i*) **Translation.** Consider $W = z + z_0$ ...(1)

$$\Rightarrow \quad (u + iv) = (x + iy) + (x_0 + iy_0)$$

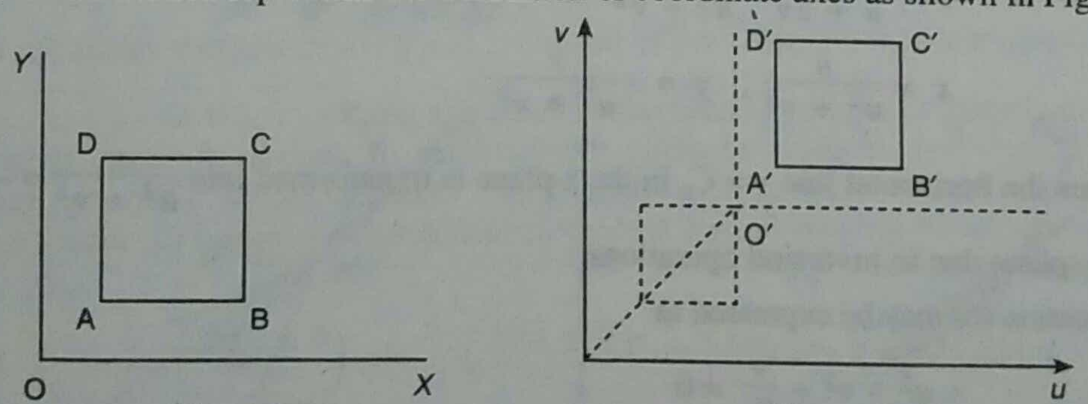
$$\Rightarrow \quad u = x + x_0 \quad \text{and} \quad v = y + y_0$$



(*a*) z – plane (*b*) w – plane

Clearly the point $P(x,y)$ in the z-plane is mapped into the point $P'(x + x_0, y + y_0)$ in the w-plane. Similarly other points of the z-plane are shifted to w-plane through a vector z_0.

Thus a pure translation represents the translation of coordinate axes as shown in Fig.



(*a*) z – plane (*b*) w – plane

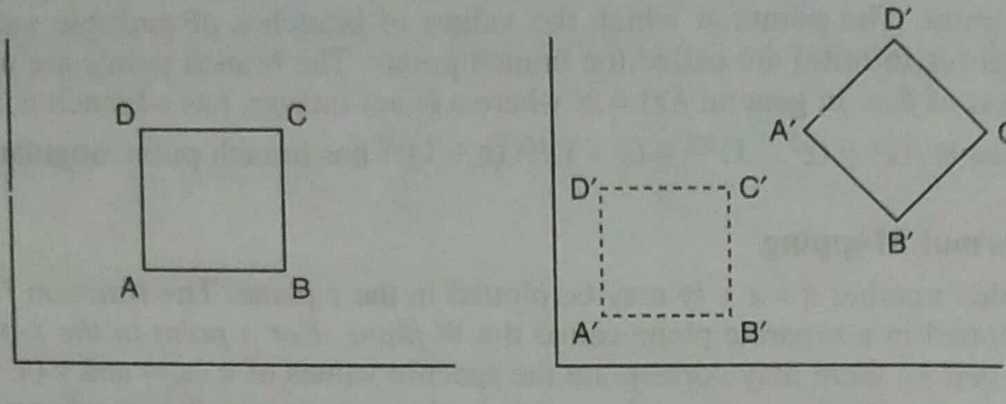
(*ii*) **Rotation.** Consider $w = z\,z_0$ ...(2)

where $z_0 = x_0 + iy_0$ is fixed complex

In polar coordinates, by taking $w = Re^{i\phi}$, $z = re^{i\theta}$, $z_0 = r_0e^{i\theta_0}$, equation (2) may be expressed as

$$Re^{i\phi} = rr_0\, e^{i(\theta + \theta_0)}$$

Modulus $R = rr_0$, argument $\phi = 0 + 0_0$ then the transformation $w = zz_0$ corresponds to a multiplication of modulus by r_0 and increase of argument by θ_0. Thus this mapping is equivalent to rotation of the complex variable through an angle θ_0.



(*a*) z – plane (*b*) w – plane Rotation

(*iii*) **Inversion.** It is represented by $w = \frac{1}{z}$

In polar coordinates it may be expressed as $Re^{2\phi} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$

$$\Rightarrow \qquad R = \frac{1}{r} \text{ and } \phi = -\theta$$

which shows the *inversion*. Under this transformation, the interior of the circle in *z*-plane is mapped into exterior in the *w*-plane and vice versa. Due to change in sign of the argument, this transformation involves of the *y*-axis exactly like the complex conjugation.

We now show how the points in *z*-plane get transformed into *w*-plane under inversion. The inversion operations $w = \frac{1}{2}$ ...(3)

may be expressed as

$$u + iv = \frac{1}{x + 2y} = \frac{x - iy}{x^2 + y^2}$$

$$\Rightarrow \qquad u = \frac{x}{x^2 + y^2} \text{ and } v = -\frac{y}{x^2 + y^2} \qquad ...(4)$$

$$\text{Also} \quad x + iy = \frac{1}{u + 2v} = \frac{u - iv}{u^2 + v^2}$$

$$\Rightarrow \qquad x = \frac{u}{u^2 + v^2}, \quad y = -\frac{v}{u^2 + v^2} \qquad ...(5)$$

Thus the horizontal line $y = C_p$ in the *z*-plane is transformed into $\frac{v}{u^2 + v^2} = -C_1$, ...(6)

in the *w*-plane due to inversion operations.

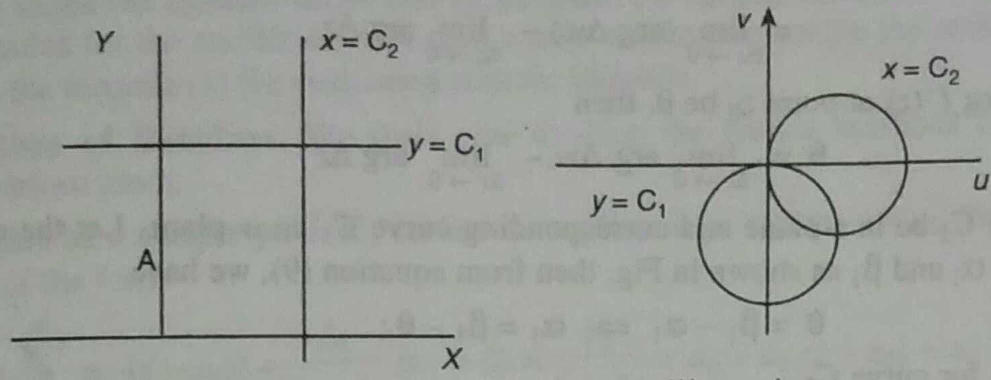
Equation (6) may be expressed as

$$u^2 + v^2 + \frac{v}{C_1} = 0$$

$$\Rightarrow \qquad u^2 + v^2 + \frac{V}{C_1} + \frac{1}{4C_1^2} = \frac{1}{uC_1^2}$$

$$\Rightarrow \qquad u^2 + \left(v + \frac{1}{2C_1}\right)^2 = \left(\frac{1}{2C_1}\right)^2 \qquad ...(7)$$

this equation represents a circle of radius $\frac{1}{2C_1}$ and centre at $\left(0, -\frac{1}{2C_1}\right)$ in the w-plane. Thus horizontal line in the z-plane is transferred into a circle in the w-plane (Fig.)



(a) z – plane (b) w – plane

Similarly a vertical line $n = C_2$ in z-plane, is transformed into a circle of radius $(1/2C_2)$ and centre at $\left(\frac{1}{2C_2}, 0\right)$ in the w-plane.

Now consider a circle of radius r with centre at origin 0 in the z-plane *i.e.*, $x^2 + y^2 = r^2$ Substituting value of x and y from equation (5), we get

$$\Rightarrow \quad \frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} = r^2$$

$$\Rightarrow \quad u^2 + v^2 = \frac{1}{r^2} = a^2$$

which represents a circle of radius $a = \frac{1}{r}$ in the w-plane with centre at origin.

Definition Conformed Transformation

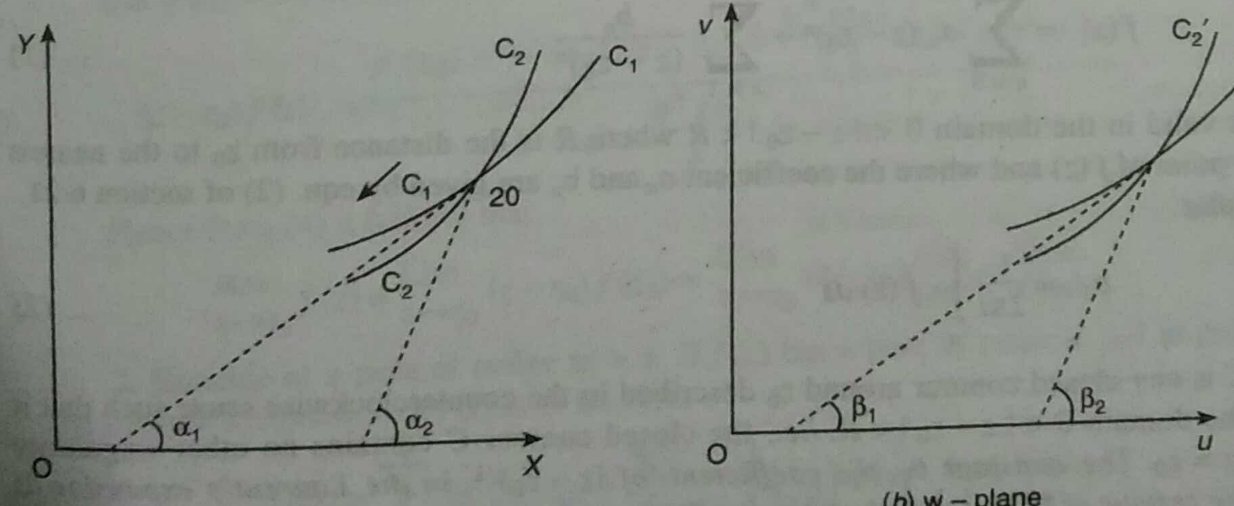
A mapping is said to be conformed if the angle between two curves at any point in the z-plane is preserved in magnitude and sense by the mapping in the w-plane.

Theorem : *If the function $f(z)$ is analytic, the mapping is conformed.*

Proof : Let the function $w = f(z)$ be analytic, then analyticity may be represented by constancy of real and imaginary points at a given point (z_0).

For $W = 1(z)$ to be analytic, it is single valued and differentiable

$$\frac{dw}{dz} = \frac{df}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} \qquad \ldots(8)$$



(a) z – plane (b) w – plane

If this equation is expressed in polar form, then we may equate modulus and argument, *i.e.*,

$$\arg f'(z) = \arg_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} (\arg \Delta w - \arg \Delta z)$$

$$= \lim_{\Delta z \to 0} (\arg \Delta w) - \lim_{\Delta z \to 0} \arg \Delta z$$

then let arg $f'(z)$ at point z_0 be θ, then

$$\theta = \lim_{\Delta z \to 0} \arg \Delta w - \lim_{\Delta z \to 0} \arg \Delta z \quad \ldots(9)$$

Let curve C_1 be in z-plane and corresponding curve C_1' in w-plane. Let the arguments of Δz and Δw be α_1 and β_1 as shown in Fig. then from equation (9), we have

$$\theta = \beta_1 - \alpha_1 \Rightarrow \alpha_1 = \beta_1 - \theta$$

Similarly for curve C_2

$$\alpha_2 = \beta_2 - \theta \text{ (as } \theta \text{ at point } z_0 \text{ is fixed)}$$

Then angle between two tangents

$$\alpha_2 - \alpha_1 = (\beta_2 - \theta) - (\beta_1 - \theta) = \beta_2 - \beta_1$$

This proves conformed mapping.

Ex. 37. *Find the image of the circle* $| z - 1 | = 1$ *in the complex plane under the transformation* $w = \frac{1}{z}$.

Solution. $w = \frac{1}{z} \Rightarrow u + 2v = \frac{1}{n + 1y} = \frac{x - iy}{x^2 + y^2}$

$$\Rightarrow u = \frac{n}{x^2 + y^2} \quad \text{and} \quad V = -\frac{y}{x^2 + y^2}$$

Given $| z - 1 | = 1$ or $| x + iy - 1 | = 1$ or $(x - 1)^2 + y^2 = 1$

$$\Rightarrow x^2 + 1 - 2x + y^2 = 1 \Rightarrow x^2 + y^2 = 2x$$

or $$\frac{x}{x^2 + y^2} = \frac{1}{2} \Rightarrow u = \frac{1}{2}$$

$$\Rightarrow \boxed{2u - 1 = 0}$$

6·23. Residues and their Evaluation

Residues. If the function $f(z)$ has a pole or an isolated essential singularity at $z = z_0$, then $f(z)$ may be represented by a Laurent series about z_0, *i.e.*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \quad \ldots (1)$$

which is valid in the domain $0 < | z - z_0 | < R$ where R is the distance from z_0 to the nearest singular point of $f(z)$ and where the coefficient a_n and b_n are given by eqn. (2) of section 6·21. In particular

$$b_1 = \frac{1}{2\pi i} \int_C f(z)\, dz \quad \ldots (2)$$

where C is any closed contour around z_0 described in the counterclockwise sense such that it lies in the domain $0 < | z - z_0 | < R$, *i.e.*, the closed contour C contains no other singularity except $z = z_0$. *The constant* b_1, *the coefficient of* $(z - z_0)^{-1}$, *in the Laurent's expansion is called the residue of* $f(z)$ *at singularity* $z = z_0$. It is usually denoted by

$$b_1 = \operatorname*{Res}_{z = z_0} f(z) \quad \text{... (3)}$$

In many cases the Laurent series can be obtained by various methods, without using the integral formulae for the coefficients. In such a case we may determine the residue b_1 directly and then use the formula (2) for evaluating contour integrals.

Evaluation of Residues. We shall now develop the simple methods for finding the residues in various cases.

1. Residue at a simple pole. If $f(z)$ has a simple pole at a point $z = z_0$, then the Laurent series (1) is of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} (b_1 \neq 0) = \frac{b_1}{z - z_0} + a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \ldots$$

Multiplying both sides by $(z - z_0)$, we get

$$(z - z_0) f(z) = b_1 + a_0 (z - z_0) + a_1 (z - z_0)^2 + a_2 (z - z_0)^3 + \ldots$$

Taking the limit $z \to z_0$, the right hand side approaches b_1.

Hence in view of (3) we obtain for the residue of $f(z)$ at simple pole $z = z_0$

$$\operatorname*{Res}_{z = z_0} f(z) = b_1 = \lim_{z \to z_0} (z - z_0) f(z) \quad \text{...(4)}$$

In this case **another useful formula** may be obtained as follows :

If $f(z)$ has a simple pole at $z = z_0$, then $f(z)$ may be put in the fractional form

$$f(z) = \frac{p(z)}{q(z)}$$

where p and q are analytic at $z = z_0$, $p(z_0) \neq 0$; but $q(z_0) = 0$ since z_0 is a singularity of $f(z)$ if and only if $q(z_0) = 0$. As $p(z)$ and $q(z)$ are analytic at $z = z_0$, they may be expanded by Taylor's series in the neighbourhood of z_0. Then we obtain

$$f(z) = \frac{p(z_0) + p'(z_0)(z - z_0) + \dfrac{p''(z_0)}{2!}(z - z_0)^2 + \ldots}{q(z_0) + q'(z_0)(z - z_0) + \dfrac{q''(z_0)}{2!}(z - z_0)^2 + \ldots}$$

But $q(z_0) = 0$; therefore

$$(z - z_0) f(z) = \frac{p(z_0) + p'(z_0)(z - z_0) + \dfrac{p''(z_0)}{2!}(z - z_0)^2 + \ldots}{q'(z_0) + \dfrac{q''(z_0)}{2!}(z - z_0) + \ldots}$$

Hence from (4) it follows that

$$\operatorname*{Res}_{z \to z_0} f(z) = \lim_{z \to z_0} (z - z_0) f(z_0) = \lim_{z \to z_0} (z - z_0) \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)} \quad \text{... (5)}$$

2. Residue at a pole of order m > 1. If $f(z)$ has a pole of order $m > 1$ at point z_0, then the Laurent's expansion of $f(z)$ is of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0) + \sum_{n=1}^{m} \frac{b_n}{(z - z_0)^n} \quad \text{where } b_m \neq 0$$

$$= \frac{b_m}{(z-z_0)^m} + \frac{b_{m-1}}{(z-z_0)^{m-1}} + \ldots + \frac{b_2}{(z-z_0)^2} + \frac{b_1}{(z-z_0)} + a_0 + a_1 (z-z_0) + \ldots$$

Multiplying both sides by $(z-z_0)^m$, we get

$$(z-z_0)^m f(z) = b_m + b_{m-1}(z-z_0) + \ldots + b_2 (z-z_0)^{m-2} + b_1 (z-z_0)^{m-1} + a_0 (z-z_0)^m + a_1 (z-z_0)^{m+1} + \ldots \quad \ldots (6)$$

Now differentiating both sides of eqn. (6) with respect to z, $(m-1)$ times and take the limit $z \to z_0$, we obtain

$$\lim_{z \to z_0} \left[\frac{d^{m-1}}{dz^{m-1}} \{(z-z_0)^m f(z)\} \right] = b_1 (m-1)!$$

Hence *the residue b_1 at the multiple pole of order m* is

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z \to z_0} \left[\frac{d^{m-1}}{dz^{m-1}} \{(z-z_0)^m f(z)\} \right] \quad \ldots (7)$$

An alternative method for finding the residue at a pole $z = z_0$ of any order m is given below :—

If $f(z)$ has a pole of order m at point $z = z_0$, then the Laurent's expansion of $f(z)$ is of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \ldots + \frac{b_m}{(z-z_0)^m}$$

Substituting $z - z_0 = t$ *i.e.*, $z = z_0 + t$, we obtain

$$f(z_0 + t) = \sum_{n=0}^{\infty} a_n t^n + \frac{b_1}{t} + \frac{b_2}{t^2} + \ldots + \frac{b_m}{t^m} \quad \ldots (8)$$

where b_1 is the residue of $f(z)$ at $z = z_0$, the pole of any order m.

Here b_1 is the coefficient of $\frac{1}{t}$ in the expansion (8). Thus *collecting the coefficients of $\frac{1}{t}$ in the expansion of $f(z)$ after putting $z = z_0 + t$, we may obtain the residue of $f(z)$ at $z = z_0$, a pole of any order.*

3. The residue at infinity. In the theory of complex variable it is convenient to regard infinity as a single point. The behaviour of $f(z)$ *at infinity* is considered by making the substiution $z = \frac{1}{t}$ and examining $f\left(\frac{1}{t}\right)$ at $t = 0$. We then say that $f(z)$ is analytic or has a simple pole or has an essential singularity at infinity according as $f\left(\frac{1}{t}\right)$ has the corresponding property at $t = 0$.

If $f(z)$ has an isolated singularity at *infinity*, then the residue of $f(z)$ at infinity is defined as

$$\operatorname*{Res}_{z=\infty} f(z) = \frac{1}{2\pi i} \int_{-C} f(z)\, dz \quad \ldots (9)$$

where C is a large circle which encloses all other (finite) singularities of $f(z)$; $(-C)$ indicates that the integration is taken round C in the *negative* (*clockwise*) *sense* [negative with respect to origin], provided that this integral has a definite value.

If we apply the transformation $z = \frac{1}{t}$ to the integral (9) it becomes

$$\underset{z=\infty}{\text{Res}} f(z) = \underset{t=0}{\text{Res}} f\left(\frac{1}{t}\right) = \frac{1}{2\pi i}\int_{C_0} f\left(\frac{1}{t}\right)\left(-\frac{dt}{t^2}\right) = \frac{1}{2\pi i}\int_{C_0}\left\{-\frac{f\left(\frac{1}{t}\right)}{t^2}\right\} dt$$

where C_0 is a small circle described about origin in the counterclockwise (positive) sense.

Therefore, the residue of $f\left(\frac{1}{t}\right)$ at $t = 0$

$$= \lim_{t\to 0}\left[t.\left\{-\frac{f\left(\frac{1}{t}\right)}{t^2}\right\}\right] = \lim_{t\to 0}\left[-\frac{1}{t} f\left(\frac{1}{t}\right)\right]$$

provided this limit has a definite value.

Hence the residue of $f(z)$ at $z = \infty$ is

$$\underset{z=\infty}{\text{Res}} f(z) = \lim_{z\to\infty}\left[-z f(z)\right] \qquad \text{... (10)}$$

provided this limit has a definite value.

Alternative method for finding residue at infinity. If the $\lim_{z\to\infty}[-z f(z)]$ does not exist, then we may find the residue of $f(z)$ at infinity by means of following theorem :

Theorem 6·13. *The residue at infinity is the negative of the coefficient of* $\frac{1}{z}$ *in the expansion of* $f(z)$ *for values of* z *in the neighbourhood of* $z = \infty$.

Proof. Let $f(z)$ have a pole of order m at infinity. Then $f\left(\frac{1}{z}\right)$ has a pole of the order m at $z = 0$. As such we may expand $f\left(\frac{1}{z}\right)$ in a Laurent's series for values of z in the annulus $0 < |z| < \rho$ is very small, that is

$$f\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{m} b_n \left(\frac{1}{z}\right)^n$$

Replacing z by $\frac{1}{z}$, the expansion of $f(z)$ in the neighbourhood of $z = \infty$ is given by

$$f(z) = \sum_{n=1}^{m} b_n z^n + \sum_{n=0}^{\infty} a_n z^{-n} \qquad \text{... (11)}$$

So the residue of $f(z)$ at infinity by definition, is

$$-\frac{1}{2\pi i}\int_C f(z)\, dz = -\frac{1}{2\pi i}\left[\int_C \sum_{n=1}^{m} b_n z^n\, dz + \int_C \sum_{n=0}^{\infty} a_n z^{-n}\, dz\right] = -\frac{1}{2\pi i}\int_C \frac{a_1}{z}\, dz\,;$$

Obviously, the function $\phi(z) = f''(z)/f'(z)$ has a simple pole at $z = z_0$. The residue of $\phi(z)$ at $z = z_0$ is given by

$$\underset{z=z_0}{\text{Res}}\,\phi(z) = \lim_{z\to z_0} (z - z_0)\,\phi(z)$$

$$= \lim_{z\to z_0} \frac{(z-z_0)^2 p''(z) - 2n(z-z_0)p'(z) + n(n+1)p(z)}{(z-z_0)p'(z) - n\,p(z)}$$

$$= [n(n+1)p(z_0)/\{-n\,p(z_0)\} = -(n+1)$$

6·24. Cauchy Residue Theorem

Theorem 6·14. *If a single-valued function f (z) is analytic within and on a closed contour C, except for a finite number of singular points $z_1, z_2 \ldots z_n$ interior to C, then*

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \underset{z=z_k}{\text{Res}} f(z) \qquad \ldots (1)$$

$= 2\pi i \times$ (sum of residues at all singularities within C)

where the integral is taken in the counterclockwise sense around C.

Proof. If a function has only a finite number of singular points in a given domain, then those singular points are necessarily isolated.

Let a single-valued function $f(z)$ be analytic within and on a closed counter C except for a finite number of singular points $z_1, z_2, \ldots z_n$ interior to C. Let us enclose each of the singular points z_k in a small circle C_k such that these n circles and the contour C are all separated (Fig. 6·14). These circles together with the curve C form the boundary of a multiple connected domain D within and on which $f(z)$ is analytic. Now deforming the contour C by cross-cuts as shown in fig. 6·17, the Cauchy's integral formula extended to such regions leads to

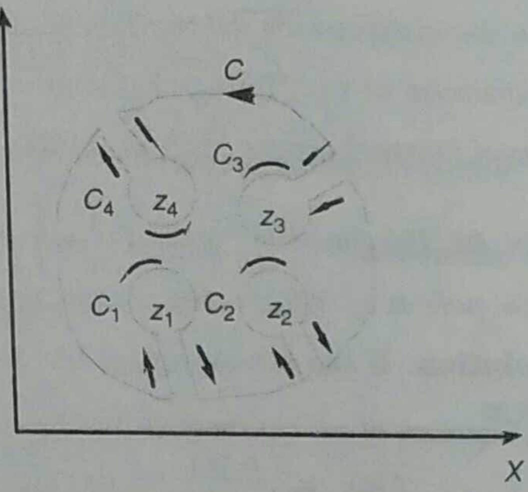


Fig. 6·17

$$\int_C f(z)\,dz - \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz - \ldots - \int_{C_n} f(z)\,dz = 0$$

where all integrals are taken in counterclockwise sense.

$$\therefore \quad \int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \ldots + \int_{C_n} f(z)\,dz \qquad \ldots (2)$$

But $\underset{z=z_k}{\text{Res}} f(z) = \dfrac{1}{2\pi i}\displaystyle\int_{C_k} f(z)\,dz$ i.e., $\displaystyle\int_{C_k} f(z)\,dz = 2\pi i \underset{z=z_k}{\text{Res}} f(z)$ $\ldots (3)$

Therefore equation (2) gives

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \underset{z=z_k}{\text{Res}} f(z) \quad \text{...(4)}$$

$$= 2\pi i \times (\text{sum of residues at all singular points within } C).$$

This proves Cauchy residue theorem. It is of extreme importance in evaluating complex and real integrals.

Cor. *If a function is analytic in the whole domain except at a finite number of singular points including that at infinity, then the sum of residues at these singularities (including that at infinity) is zero.* This can be proved as follows :

Let C denote the boundary of the domain enclosing all the singularities of analytic function $f(z)$ except at infinity, then according to Cauchy Residue theorem

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \underset{z=z_k}{\text{Res}} f(z)$$

where $\sum_{k=1}^{n} \underset{z=z_k}{\text{Res}} f(z)$ denotes the sum of all residues at all the finite singularities enclosed by C.

i.e.,

$$\sum_{k=1}^{n} \underset{z=z_k}{\text{Res}} f(z) = \frac{1}{2\pi i}\int_C f(z)\,dz \quad \text{... (5)}$$

But by definition the residue at infinity

$$\underset{z=\infty}{\text{Res}} f(z) = \frac{1}{2\pi i}\int_{-C} f(z)\,dz = -\frac{1}{2\pi i}\int_C f(z)\,dz \quad \text{... (6)}$$

Adding (5) and (6), we obtain

The *sum of all residues of* $f(z)$

$$\sum_{k=1}^{n} \underset{z=z_k}{\text{Res}} f(z) + \underset{z=z_k}{\text{Res}} f(z) = 0 \quad \text{... (7)}$$

6·25. Evaluation of Definite Integrals

By the use of *Cauchy Residue Theorem* certain type of definite integrals may be evaluated. In each given case the choice of a suitable curve along which the integration is to be affected plays an important role. This curve is usually known as *contour* and the integration along the same is called *Contour Integration.* It may be observed that a definite integral that can be evaluated by the use of Cauchy residue theorem may be evaluated by other methods ; although not so easily.

Two Important Theorems. In the evaluation of integrals by the method of complex variables, it is sometimes convenient to use the following theorems.

Theorem 6·15. *If AB is an arc of a circle* $|z| = R$ having $\theta_1 \le \theta \le \theta_2$ *and* $\lim_{R\to\infty} z f(z)$ *tends uniformly to b* ; then

$$\lim_{R\to\infty} \int_{BA} f(z)\, dz = ib\,(\theta_2 - \theta_1) \qquad \dots (1)$$

Proof. Consider an arc of a circle $|z| = R$ having $\theta_1 \le \theta \le \theta_2$. Given $\lim_{R\to\infty} z f(z) \to b$

Choosing the radius of the circle $|z| = R$ sufficiently great, we can make

$$|z f(z) - b| < \epsilon, \text{ where } \epsilon \text{ is a very small number.}$$

This implies

$$z f(z) = b + \eta \text{ where } |\eta| < \epsilon$$

i.e.,

$$f(z) = \frac{b + \eta}{z}$$

$\therefore$

$$\int_{AB} f(z)\, dz = \int_{AB} \frac{b + \eta}{z}\, dz$$

Substituting $z = Re^{i\theta}$; so that $dz = Re^{i\theta}\, id\theta$, we get

$$\int_{AB} f(z)\, dz = \int_{\theta_1}^{\theta_2} (b + \eta)\, id\theta = ib \int_{\theta_1}^{\theta_2} d\theta + \int_{\theta_1}^{\theta_2} \eta i d\theta$$

i.e.,

$$\int_{AB} f(z)\, dz = ib\,(\theta_2 - \theta_1) + \int_{\theta_1}^{\theta_2} \eta i d\theta$$

$$\therefore \left| \int_{AB} f(z)\, dz - ib\,(\theta_2 - \theta_1) \right| \le \int_{\theta_1}^{\theta_2} |\eta|\,|id\theta| < \epsilon\,(\theta_2 - \theta_1)$$

In the limit $R\to\infty$, $\epsilon \to 0$, therefore we have

$$\lim_{R\to\infty} \int_{AB} f(z)\, dz - ib\,(\theta_2 - \theta_1) = 0$$

i.e.,

$$\lim_{R\to\infty} \int_{AB} f(z)\, dz = ib\,(\theta_2 - \theta_1)$$

Theorem 6·16. *If AB is the arc of a circle $|z - z_0| = r$, having $\theta_1 \le \theta \le \theta_2$ and $\lim_{z\to z_0} (z - z_0) f(z) = b$; where b is a constant ; then*

$$\lim_{r\to 0} \int_{AB} f(z)\, dz = ib\,(\theta_2 - \theta_1) \qquad \dots (2)$$

Proof. Consider an arc of circle $|z - z_0| = r$ having $\theta_1 \le \theta \le \theta_2$.

Given $\lim_{z\to z_0} (z - z_0) f(z) = b$.

Choosing the radius of the circle $|z - z_0| = r$ sufficiently small, we can make

$$|(z - z_0) f(z) - b| < \epsilon \text{ for } |z - z_0| < \delta$$

where δ is very small. Choosing ϵ less than δ, we can write

$$(z - z_0) f(z) = b + \eta \text{ where } |\eta| < \epsilon$$

i.e. $$f(z) = \frac{b + \eta}{z - z_0}$$

$$\therefore \quad \int_{AB} f(z)\, dz = \int_{AB} \frac{b + \eta}{z - z_0} dz$$

Putting $(z - z_0) = re^{i\theta}$ i.e., $dz = re^{i\theta}\, id\theta$, we have

$$\int_{AB} f(z)\, dz = \int_{\theta_1}^{\theta_2} \frac{(b + \eta)\, re^{i\theta}\, id\theta}{re^{i\theta}} = bi \int_{\theta_1}^{\theta_2} d\theta + \int_{\theta_1}^{\theta_2} \eta i d\theta$$

$$= bi\,(\theta_2 - \theta_1) + \int_{\theta_1}^{\theta_2} \eta i d\theta$$

This gives

$$\left| \int_{AB} f(z)\, dz - ib\,(\theta_2 - \theta_1) \right| \le \int_{\theta_1}^{\theta_2} |\eta|\,|i|\,|d\theta| < \epsilon\,(\theta_2 - \theta_1)$$

In the limit $r \to 0$, $\epsilon \to 0$; therefore

$$\lim_{r \to 0} \int_{AB} f(z)\, dz = ib\,(\theta_2 - \theta_1)$$

6·25. (a) Define Integrals of Trigonometric Functions of cos θ and sin θ ; Integration Round the Unit Circle

Let us first apply the method of contour integration to the definite integrals of the type

$$\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta \qquad \text{... (1)}$$

where the integrand is a real radial function of $\cos\theta$ and $\sin\theta$ that is finite in the range of integration.

Let us apply the transformation $z = e^{i\theta}$ ... (2)

Then we obtain $$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$\sin\theta = \frac{e^{i\theta} - e^{i\theta}}{2i} = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$$dz = ie^{i\theta}\, d\theta \text{ so that } d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz}$$

and so the integral (1) takes the form

$$\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta = \int_C F\left[\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)\right]\frac{dz}{iz} = \int_C f(z)\, dz \text{ (say)}$$

We thus see that the integrand becomes a rational function $f(z)$ of z. As θ changes from 0 to 2π, the variable z ranges around the unit circle $|z| = |e^{i\theta}| = 1$ in the counterclockwise sense. This latter integral can be evaluated by means of residue theorem so that

$$\int_C f(z)\, dz = 2\pi i \sum_k \underset{z=z_k}{\text{Res}} f(z) \qquad \text{... (3)}$$

where $\sum_k \underset{z=z_k}{\text{Res}} f(z)$ denotes the sum of residues of $f(z)$ at its poles z_k inside the circle C, $|z| = 1$.

Ex. 45. *Apply calculus of residues to show that*

(*i*) $\displaystyle\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}, \; a > b > 0.$ (*Bharatidasan, 1989, Nagpur 1999, 96*)

(*ii*) $\displaystyle\int_0^{2\pi} \frac{d\theta}{1 + \in\cos\theta} = \frac{2\pi}{\sqrt{1 - \in^2}}, \; 0 \le \in < 1$ (*Delhi 2006, Rohilkhand 1997*)

(*iii*) $\displaystyle\int_0^{2\pi} \frac{d\theta}{25 - 24\cos\theta} = \frac{2\pi}{7}$ (*Meerut 1992*)

(*iv*) $\displaystyle\int_0^{2\pi} \frac{d\theta}{2 + \cos\theta} = \frac{2\pi}{\sqrt{3}}$ (*Pondicherry 1993, Rohilkhand 1989, Meerut 1985*]

(*v*) $\displaystyle\int_0^{2\pi} \frac{d\theta}{5 + 4\cos\theta} = \frac{2\pi}{3}$ (*Meerut 2008, 2006, 2004*)

Solution. (*i*) Let $I = \displaystyle\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta}$

Putting $z = e^{i\theta}$; so that $d\theta = \dfrac{dz}{iz}$ and $\cos\theta = \dfrac{1}{2}\left(z + \dfrac{1}{z}\right)$; we get

$$I = \int_C \frac{dz/iz}{\left[a + \frac{b}{2}\left(z + \frac{1}{z}\right)\right]} \quad \text{where } C \text{ denotes the unit circle } |z| = 1$$

i.e.,

$$I = \frac{2}{ib}\int_C \frac{dz}{z^2 + \frac{2a}{b}z + 1} = \int_C f(z)\, dz \qquad \text{... (1)}$$