Chapter 1

Electrostatics in Vacuum

1.1 Electric Charge

Electrostatics is a branch of physics, which deals with the behaviour of *electric charges* at rest. So it is important to seek the answer to the question: What is electric charge? Many substances such as ebonite, glass, rubber, etc. when rubbed with silk, flannel or other suitable substances acquire the property of attracting light bodies like bits of paper. The substances in such state are said to be charged and they are said to possess what is called *electric charge*. Simple experiments show that there are two types of charges. One variety is called *positive charge* and the other negative charge. Like charges are found to repel each other and unlike charges attract each other. According to our present knowledge of the atomic structure of matter, all elements consist of positively charged protons, negatively charged electrons and neutral neutrons. The charge of a proton is numerically equal to that of the electron. In an atom or an ordinary piece of matter there are equal number of protons and electrons. As a result, matter as a whole is electrically neutral. Whenever two different bodies are rubbed with each other some of the electrons from one body may be transferred to the other body. The body which gets excess electrons becomes negatively charged and the body which loses electrons becomes positively charged. Matter as a whole or elementary particles of matter like protons, electrons can carry charge. But it is found that electric charge cannot exist without matter. Charge is thus a fundamental and characteristic property of elementary particles. It is a scalar quantity. It adds up like real numbers. While adding charges one must take care of their signs also.

Quantization of charge

In nature all electric charges are found to be integral multiples of one basic unit, the magnitude of electronic charge (e). This occurrence of electric charge in discrete units is called the *quantization of charge*. Since the electrification of a body is generally caused by taking up a number of electrons or by giving up a number of electrons the charge on the body will be equal to an integral multiple of electronic charge. Experiments show that the magnitude of charge of all elementary charged particles is equal to the magnitude of electronic charge. Very recently the existence of elementary particles (called *quarks*) with fractional electronic charge has been suggested theoretically. However, its existence in free state has not yet been proved experimentally.

In the macroscopic level the charge we encounter in *charged bodies* is so large compared with electronic charge that to a good approximation we can regard charge as continuous.

Conservation of charge

Total charge i.e., the algebraic sum of positive and negative charges in an isolated system remains constant for all times. This is known as the law of conservation of charge. Here by *isolated system* we mean that no charge is allowed to cross the boundary of the system. The law of conservation of charge allows simultaneous *creation* or *annihilation* of equal and opposite amount of charges within the isolated system. For example, a high energy neutral photon may produce an electron of charge $-e$ and a positron of charge $+e$ or an electron-positron pair may combine to produce a photon. This does not violate the law of conservation of charge. But a single positive charge or a single negative charge can neither be created nor destroyed.

1.2 Coulomb's Law

From an experimental study Coulomb in 1785 discovered a law of force between charge particles. This is known as *Coulomb's law*. According to this law *the force of interaction between two stationary point charges is proportional to the product of the charges, inversely proportional to the square of the distance between them and it acts along the line joining the charges.*

Suppose $\vec{r}_1$ and $\vec{r}_2$ be the position vectors of two point charges q_1 and q_2 with respect to some coordinate system as shown in Fig 1.2-1.

Let $\vec{r}_{21} = \vec{r}_2 - \vec{r}_1$ be the vector from q_1 to q_2, $r_{21} = |\vec{r}_{21}|$ be the distance between q_1 and q_2 and $\hat{r}_{21} = \vec{r}_{21}/|\vec{r}_{21}|$ be the unit vector in the direction from q_1 to q_2. Then from Coulomb's law the force on q_2 due to q_1 is

$$\vec{F}_{21} = k \cdot \frac{q_1 q_2}{r_{21}^2} \hat{r}_{21},$$

where k is a proportionality constant, its value depends on the nature of surrounding medium and the system of units used. In vacuum in SI units F_{21} is in newtons (N), q_1, q_2 in coulombs (C), r_{21} in metres and $k = \frac{1}{4\pi\epsilon_0}$. Here ϵ_0 is called the *permittivity of*

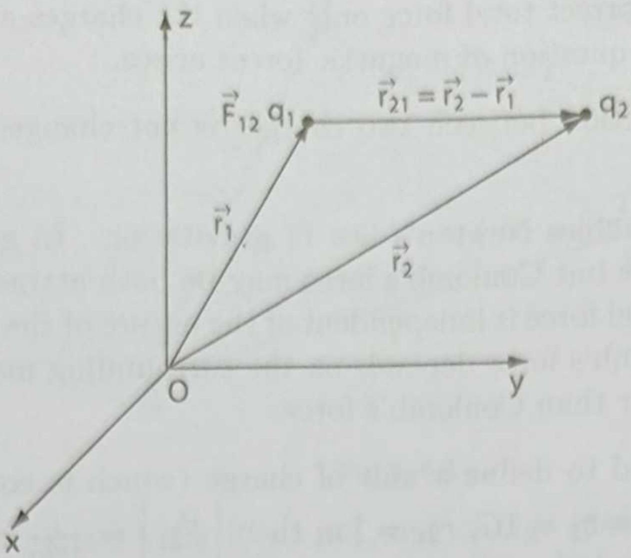


Fig 1.2-1

free space. The measured value of ϵ_0 is 8.85×10^{-12} $C^2/N\text{-}m^2$. For simplified numerical calculations one can approximately use

$$\frac{1}{4\pi\epsilon_0} = 9 \times 10^9 \text{ N-m}^2/\text{C}^2.$$

Thus the force on q_2 due to q_1 is

$$\vec{F}_{21} = \frac{1}{4\pi\epsilon_0} \cdot \frac{q_1 q_2}{r_{21}^2} \hat{r}_{21} \tag{1.2-1}$$

The force on q_1 due to q_2 is

$$\vec{F}_{12} = \frac{1}{4\pi\epsilon_0} \cdot \frac{q_2 q_1}{r_{12}^2} \hat{r}_{12},$$

where $\hat{r}_{12}$ is the unit vector in the direction from q_2 to q_1.

So $\hat{r}_{12} = -\hat{r}_{21}$ and hence,

$$\vec{F}_{12} = -\vec{F}_{21}.$$

Thus Coulomb force is consistent with Newton's third law of motion.

Discussions

(*i*) Coulomb's law is strictly applicable to *point charges*. Practically the separation between the charges should be large compared with their spatial dimensions. Coulomb's law is found to be valid for any arbitrarily large distance between the charges. We have no experimental evidence or any theoretical reasoning to expect a breakdown of this law at large distances. However, in the domain of very small distances there is experimental evidence to believe that the law breaks down for distances of the order of a nuclear diameter, i.e., 10^{-15} m.

(*ii*) The law gives the correct total force only when the charges are at rest. In case of moving charges the question of magnetic forces arises.

(*iii*) The force of interaction between two charges is not changed by the presence of other charges.

(*iv*) Coulomb's law resembles Newton's law of gravitation. In gravitation the forces are always attractive but Coulomb's force may be both attractive and repulsive in nature. Gravitational force is independent of the nature of the medium between the particles but Coulomb's force depends on the surrounding medium. Gravitational force is much weaker than Coulomb's force.

(*v*) The law can be used to define a unit of charge (which is coulomb in SI). In Eq. (1.2-1) if we put $q_1 = q_2 = 1C$, $r_{21} = 1\,\text{m}$ then $\left|\vec{F}_{21}\right| = \frac{1}{4\pi\epsilon_0} = 9 \times 10^9 N$. Thus 1C is the charge which when placed at a distance of 1 m from another similar charge in vacuum experiences an electrostatic force of 9×10^9 N.

1.3 Principle of Superposition

The total force on a charge due to a number of other charges is given by the vector sum of the Coulomb forces exerted on the charge due to each of the other charges acting separately. This fact is called the principle of superposition. Thus if we have N point charges $q_1, q_2, q_3, \ldots, q_N$ the total force acting on a charge q_0 due to all these N charges can be found out by calculating separately the forces $\vec{F}_{01}, \vec{F}_{02}, \ldots, \vec{F}_{0N}$ on q_0 due to $q_1, q_2, \ldots, q_N$, respectively and then taking their vector sum. So total force on q_0 due to the N-charges would be

$$\vec{F}_0 = \vec{F}_{01} + \vec{F}_{02} + \cdots + \vec{F}_{0N} = \sum_{j=1}^{N} \vec{F}_{0j}.$$

1.4 Electric Field

Suppose we have a point charge Q at the position $\vec{r}'$ and another small point charge q placed at the position $\vec{r}$ with respect to some origin O (Fig 1.4-1). We are to find the force on q due to Q. Here Q is called the *source charge* and q is called *test charge*. From Coulomb's law this force is

$$\vec{F} = \frac{1}{4\pi\epsilon_0} \cdot \frac{Q \cdot q}{R^2} \hat{R} = q \cdot \vec{E}, \tag{1.4-1}$$

where

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \cdot \frac{Q}{R^2} \hat{R}; \tag{1.4-2}$$

R is the distance between Q and q, $\hat{R}$ = unit vector in the direction from Q to q.

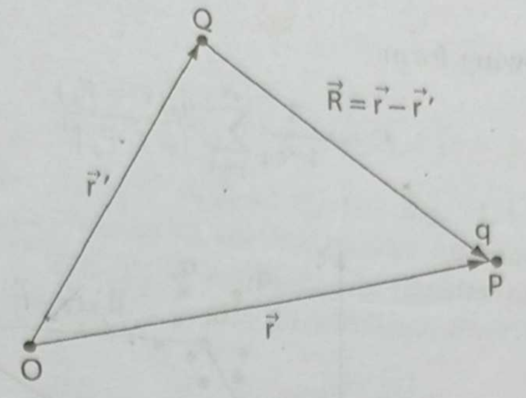


Fig 1.4-1

Obviously $\vec{E} = \vec{F}/q$, i.e., the force per unit charge placed at the point P. $\vec{E}$ is a function of the position vector $\vec{R}$ of P with respect to the source charge Q. It depends on Q but is independent of the test charge q. Thus in the region of space surrounding the source charge we can uniquely specify a vector physical quantity $\vec{E}$ at every point. This means that $\vec{E}$ defines a vector field.

Thus the force $\vec{F}$ on q due to Q may be considered as arising in two steps. (i) The charge Q sets up an electrical environment, called the *electric field* $\vec{E}$, in the surrounding space. (ii) When a charge q is placed at any point without disturbing the position of Q it experiences a force, which equals the charge q multiplied by $\vec{E}$ at that point.

From Eq. (1.4-2) *electric field at a point in space due to a charge Q may be defined as the force on a unit positive charge placed at that point.* The unit of $\vec{E}$ is N.C^{-1}. A unit charge may be high enough to disturb the configuration of the charge producing the field. A way out of this difficulty is to choose the test charge q negligibly small. Then we may define the electric field at a point as the limiting force per unit charge placed at that point, i.e.,

$$\vec{E} = \lim_{q\to 0} \frac{\vec{F}}{q}.$$

Note that $q \to 0$ is an idealisation because $q \to 0$ contradicts the quantization of charge. In case of immobile source charges we need not require this idealisation.

Our definition of electric field in Eq. (1.4-2) assumes that the source of the field is a point charge Q. If the source is a set of discrete point charges $q_1, q_2, \ldots, q_N$ whose position vectors with respect to the origin O of some coordinate system are respectively $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N$ then the electric field at any point $P(\vec{r})$ can be calculated by using the principle of superposition as

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \cdots + \vec{E}_N = \sum_{j=1}^{N} \vec{E}_j = \frac{1}{4\pi\epsilon_0} \sum_{j=1}^{N} \frac{q_j}{R_j^2} \hat{R}_j, \qquad (1.4\text{-}3)$$

where R_j is the distance of the point of observation P from the location of the point charge q_j and $\hat{R}_j$ is the unit vector in the direction from q_j to P. Equation (1.4-3) can also be put in the following form:

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\sum_{j=1}^{N}\frac{q_j(\vec{r}-\vec{r}_j)}{|\vec{r}-\vec{r}_j|^3} \tag{1.4-4}$$

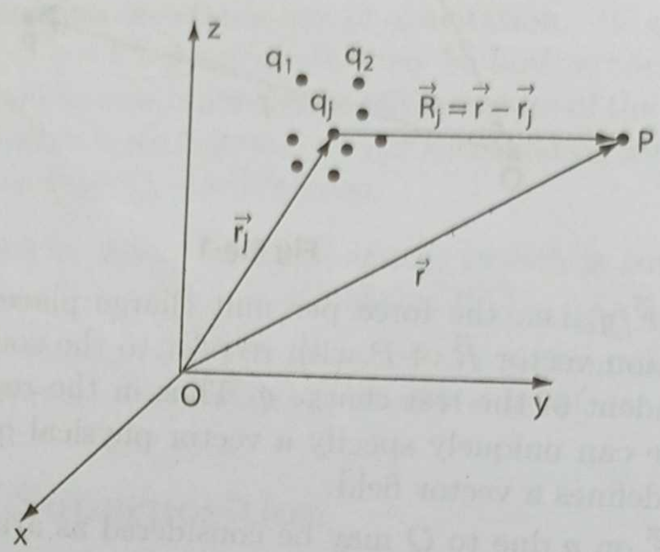


Fig 1.4-2

Continuous charge distribution

In macroscopic physics the charge we encounter is so large compared with the basic unit of charge that to a good approximation we can neglect the quantized-nature of electric charge and consider it as a continuous distribution. In a macroscopic charge distribution even a small volume element contains large number of elementary charges. In such situations it is impractical to account for these charges individually and we introduce, for convenience, the idea of charge densities. If charge is distributed over a volume V then we talk about volume density of charge. It is defined as the limiting charge per unit volume. If Δq be the amount of charge in a small volume element ΔV about the point $\vec{r}$ then volume density of charge at this point is defined as

$$\rho = \lim_{\Delta V \to 0}\frac{\Delta q}{\Delta V} = \frac{dq}{dV}.$$

Sometimes charges spread over a surface and it is described by surface charge density. It is defined as the limiting charge per unit area. If Δq be the amount of charge in a small surface element dS about a point then surface charge density at this point is defined as

$$\sigma = \lim_{\Delta S \to 0}\frac{\Delta q}{\Delta S} = \frac{dq}{dS}.$$

Similarly if charges spread out along a line we describe it by line charge density (λ). This is defined as the limiting charge per unit length. Thus if Δq be the charge on a line element Δl then line charge density

$$\lambda = \lim_{\Delta l \to 0} \frac{\Delta q}{\Delta l} = \frac{dq}{dl}.$$

Let us see that the electric field $\vec{E}$ defined by the Eq. (1.4-3) for a set of discrete point charges can be easily generalised to various charge distributions. Suppose charge is distributed over a volume V and $\rho(\vec{r}')$ is the charge density at the point $\vec{r}'(x', y', z')$. Then electric field $\vec{E}$ at the point $P(\vec{r})$ due to this volume distribution of charge can be written as

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V \hat{R} \frac{\rho(\vec{r}')\, dV}{R^2}, \tag{1.4-5}$$

where R is the distance of the point of observation P from the volume element dV $(= dx'dy'dz')$ at $\vec{r}'$ and $\hat{R} = \vec{R}/R$ is the unit vector in the direction from the volume element to the point P (Fig 1.4-3);

$$\vec{R} = \vec{r} - \vec{r}' = \hat{i}(x - x') + \hat{j}(y - y') + \hat{k}(z - z').$$

In a similar way electric field due to a surface distribution of charge (Fig 1.4-4) can be written as

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_S \hat{R} \cdot \frac{\sigma\, dS}{R^2} \tag{1.4-6}$$

and that due to a line charge distribution (Fig 1.4-5) can be written as

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_l \hat{R} \cdot \frac{\lambda dl}{R^2}. \tag{1.4-7}$$

At this point it is important to mention that a point charge can be described with a charge density by means of Dirac delta function δ, which has the following properties (see Appendix C):

(*i*) $\delta(\vec{r} - \vec{r}') = 0$ for $\vec{r}' \neq \vec{r}$.

(*ii*) $$\int_V \delta(\vec{r} - \vec{r}')\, dV = \begin{cases} 1 & \text{if } \vec{r}' = \vec{r} \text{ is included in } V \\ 0 & \text{otherwise.} \end{cases}$$

(*iii*) $$\int_V f(\vec{r}')\delta(\vec{r} - \vec{r}')\, dV = f(\vec{r}).$$

These properties show that the delta function is a very high singular function; it is zero everywhere except at a particular point; yet it has a nonzero integral. It is not a continuous function but is found to be very useful.

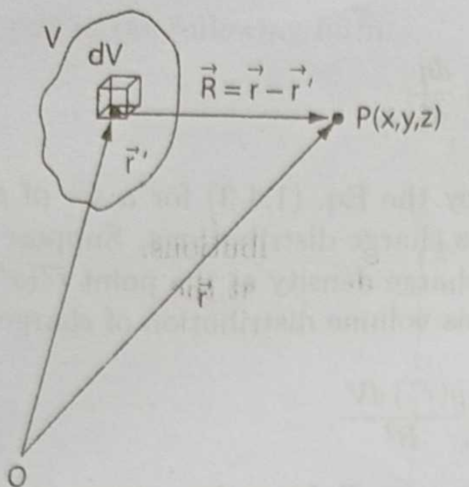


Fig 1.4-3

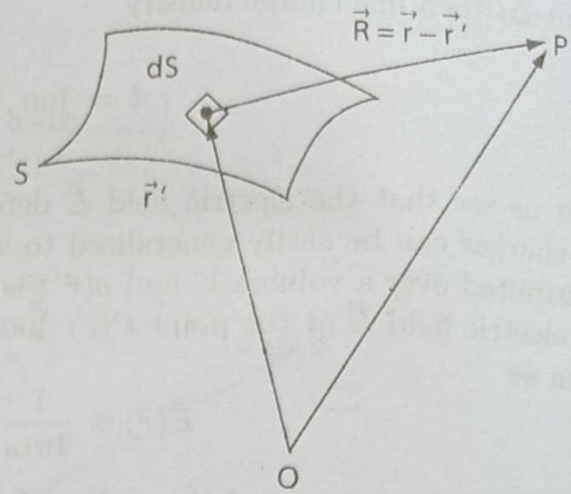


Fig 1.4-4

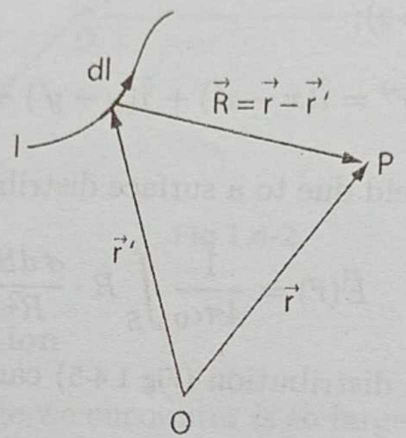


Fig 1.4-5

A point charge q_1 situated at the position $\vec{r}_1$ can be described by a volume charge density $\rho(\vec{r}\,') = q_1\delta(\vec{r}\,' - \vec{r}_1)$. This is justified because:

(i) Charge density, $\rho(\vec{r}\,') = 0$ for $\vec{r}\,' \neq \vec{r}_1$.

(ii) Total charge, $\int_V \rho(\vec{r}\,')\, dV = \int q_1\delta(\vec{r}\,' - \vec{r}_1)\, dV = q_1$.

(iii) Following Eq. (1.4-5) the electric field at the point $P(\vec{r})$ due to a point charge q_1 at $\vec{r}_1$ described by the above charge density can be calculated and shown to be identical with that calculated by considering it a point charge.

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\frac{\rho(\vec{r}\,')\, dV}{|\vec{r}-\vec{r}\,'|^2}$$

$$= \frac{1}{4\pi\epsilon_0}\int_V \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} q_1\delta(\vec{r}\,'-\vec{r}_1)\, dV$$

$$= \frac{1}{4\pi\epsilon_0}\cdot\frac{q_1(\vec{r}-\vec{r}_1)}{|\vec{r}-\vec{r}_1|^3} \quad \text{[using delta function property]}$$

Electric lines of force or field lines

An electric field associated with a particular distribution of charges can be represented geometrically by using the concept of electric lines of force. This concept was introduced by Faraday. *An electric line of force is an imaginary curve drawn in such a way that the tangent to this curve at any point gives the direction of electric field at the point.* For a single positive point charge the field lines are radially outward and for a single negative point charge they are radially inward (Fig 1.4-6). Such diagrams are also useful in representing more complicated fields. The field lines around two positive charges give a vivid pictorial description of their mutual repulsion.

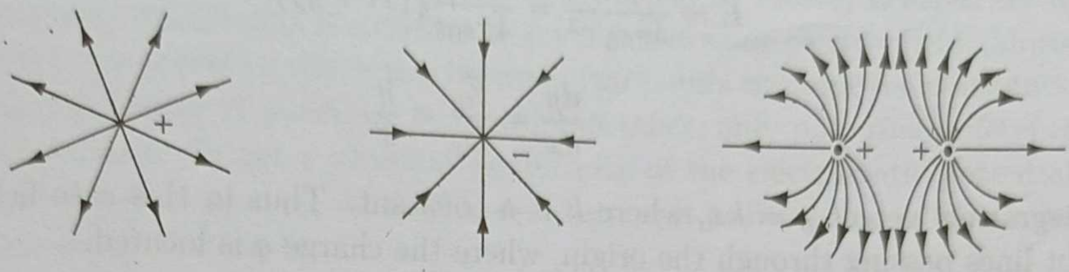


Fig 1.4-6

The magnitude of the electric field can be represented by the density of field lines, i.e., the number of lines passing through unit area perpendicular to the lines. Suppose we assume that q/ϵ_0 number of lines emanate from a charge q symmetrically in all directions. Imagine a sphere of radius r with the charge at its centre. All the lines q/ϵ_0 pass isotropically through the area $4\pi r^2$. So the number of field lines passing through unit area of this imaginary sphere would be

$$\frac{q/\epsilon_0}{4\pi r^2} = \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{r^2}.$$

This is exactly the magnitude of electric field at a distance r from a point charge.

While sketching the field lines the following properties of the field lines should be kept in mind:

(i) Field lines are continuous curves.

(ii) Two field lines cannot intersect each other.

(iii) Field lines start from a positive charge and end in a negative charge. For a single charge they can extend up to infinity.

Differential equation for lines of force

Let $\vec{r} = \vec{r}(s)$ represents a space curve. $d\vec{r}/ds$ is a vector tangent to this curve at every point. If this curve is to represent a line of force, $d\vec{r}/ds$ must be parallel to the local electric field, i.e., $d\vec{r}/ds = \alpha\vec{E}(\vec{r})$, where α is a constant.

In component form:

$$\frac{dx}{ds} = \alpha E_x, \quad \frac{dy}{ds} = \alpha E_y, \quad \frac{dz}{ds} = \alpha E_z.$$

$$\therefore \quad \frac{dx}{E_x} = \frac{dy}{E_y} = \frac{dz}{E_z}.$$

For example, let us consider the field lines of a point charge q in xy-plane. Here

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\vec{r}}{r^3} = \frac{q}{4\pi\epsilon_0 r^3}(x\hat{i} + y\hat{j}).$$

$$\therefore \quad \frac{dy}{dx} = \frac{E_y}{E_x} = \frac{y}{x}.$$

Integrating we get $y = kx$, where k is a constant. Thus in this case field lines are straight lines passing through the origin, where the charge q is located.

1.5 Electric Potential

From vector analysis it is well-known that if the curl of a vector vanishes, then the vector may be expressed as the gradient of a scalar. It is shown below that $\vec{\nabla} \times \vec{E} = 0$. This means that the electrostatic field $\vec{E}$ is conservative and it can be written as the gradient of some scaler ϕ as $\vec{E} = -\vec{\nabla}\phi$, where the negative sign is chosen for convenience. The scalar function ϕ thus introduced is known as *electrostatic potential.*

To show that $\vec{\nabla} \times \vec{E} = 0$ we first consider the electric field at the position $\vec{r}$ due a point charge q located at $\vec{r}\,'$. This is

$$\vec{E} = \frac{q}{4\pi\epsilon_0} \cdot \frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|^3}. \tag{1.5-1}$$

Now

$$\begin{aligned} \vec{\nabla} \times \vec{E} &= \frac{q}{4\pi\epsilon_0}\vec{\nabla} \times \frac{(\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3}, \quad \text{where } \vec{\nabla} \text{ operates on } \vec{r} \\ &= \frac{q}{4\pi\epsilon_0}\left\{\vec{\nabla}\frac{1}{|\vec{r} - \vec{r}\,'|^3} \times (\vec{r} - \vec{r}\,') + \frac{1}{|\vec{r} - \vec{r}\,'|^3}\vec{\nabla} \times (\vec{r} - \vec{r}\,')\right\}, \end{aligned} \tag{1.5-2}$$

where we have used the vector identity $\vec{\nabla} \times \phi\vec{A} = \vec{\nabla}\phi \times \vec{A} + \phi\vec{\nabla} \times \vec{A}$.

Now using Cartesian coordinates it can be shown by direct calculation that

$$\vec{\nabla}\frac{1}{|\vec{r} - \vec{r}\,'|^3} = -\frac{3(\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3} \quad \text{and} \quad \vec{\nabla} \times (\vec{r} - \vec{r}\,') = 0$$

Substituting these results in Eq. (1.5-2) we get

$$\vec{\nabla} \times \vec{E} = 0. \tag{1.5-3}$$

From the principle of superposition we known that the total field due to any charge distribution is given by the vector sum of the fields due to individual charges. So we can say that the result (1.5-3) holds for any charge distribution. Therefore, we can write

$$\vec{E} = -\vec{\nabla}\phi. \tag{1.5-4}$$

Since $\vec{\nabla}(\phi + \mathit{constant}) = \vec{\nabla}\phi$, the potential ϕ defined by (1.5-4) is arbitrary by some additive constant. Addition of a constant to ϕ yields the same electric field. Moreover, it does not affect the potential difference between two points because the constants cancel out. The absolute value of potential is of no importance, only potential differences have physical significance. To get a physical significance of the electrostatic potential let us calculate the work done against the field in moving a unit positive charge from a reference point a to the point b. This is

$$\begin{aligned} W_{ab} &= -\ \textit{work done by the field} \\ &= -\int_a^b \vec{E}\cdot d\vec{r} = \int_a^b \vec{\nabla}\phi\cdot d\vec{r} = \int_a^b d\phi = \phi(b) - \phi(a). \end{aligned} \tag{1.5-5}$$

Obviously the work done depends only on the positions a and b and not on the path connecting them. This indicates that the work done over any close path is zero. This result can be easily obtained from Eq. (1.5-3) by using Stokes' theorem,

$$\oint_C \vec{E}\cdot d\vec{r} = \int_S \left(\vec{\nabla}\times\vec{E}\right)\cdot d\vec{S} = 0, \tag{1.5-6}$$

where C is the contour bounding an open surface S.

This must be so because electrostatic force is conservative. Equation (1.5-5) suggests that the electrostatic potential may be considered as potential energy per unit charge. If the charge distribution that creates the electric field is localised in a finite region of space the electric field vanishes at infinite distances from the charge distribution. Then one usually takes the reference point a at infinity, where the potential is taken to be zero; $\phi(\infty) = 0$. In this case Eq. (1.5-5) gives

$$\phi(b) = -\int_\infty^b \vec{E}\cdot d\vec{r}. \tag{1.5-7}$$

Now, as the electric field is the force per unit positive charge, *the electrostatic potential at any point* (b) *may be defined as the work done by an external agency in bringing a unit positive charge from infinity to that point.*

If the same reference point is chosen for the electrostatic potential and potential energy then the potential energy of a charge is given by the product of the charge and the electric potential at the location of the charge.

Potential due to a point charge

Electric field at the position $\vec{r}$ due to a charge q located at $\vec{r}\,'$ is given by the Eq. (1.5-1). Now from vector analysis it can be shown that

$$\vec{\nabla}\frac{1}{|\vec{r}-\vec{r}\,'|} = -\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}, \quad \text{where } \vec{\nabla} \text{ operates on } \vec{r}.$$

Therefore, Eq. (1.5-1) can be rewritten as,

$$\vec{E} = -\vec{\nabla}\left[\frac{1}{4\pi\epsilon_0}\cdot\frac{q}{|\vec{r}-\vec{r}\,'|}\right]$$

Comparing it with the relation $\vec{E} = -\vec{\nabla}\phi$ we get the electrostatic potential due to the point charge as

$$\phi = \frac{1}{4\pi\epsilon_0}\cdot\frac{q}{|\vec{r}-\vec{r}\,'|}. \tag{1.5-8}$$

Potential due to charge distributions

Just like the electrostatic force ($\vec{F}$) on a test charge and the electrostatic field ($\vec{E}$), the electrostatic potential (ϕ) also obeys the principle of superposition. According to the superposition principle total force on a test charge q due to a charge distribution is given by the sum of the forces due to individual charges. Symbolically,

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \cdots$$

Dividing throughout by q we get

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \cdots$$

$$\text{or}\quad -\vec{\nabla}\phi = -\vec{\nabla}\phi_1 - \vec{\nabla}\phi_2 - \cdots$$

$$\text{or}\quad \vec{\nabla}\phi\cdot d\vec{r} = \vec{\nabla}\phi_1\cdot d\vec{r} + \vec{\nabla}\phi_2\cdot d\vec{r} + \cdots$$

Integrating this equation from ∞ to the position $\vec{r}$ we get

$$\phi = \phi_1 + \phi_2 + \cdots$$

Thus the potential at any point is the sum of the potential due to individual charges. So using Eq. (1.5-8) and this principle it is possible to calculate potential due to arbitrary charge distributions. Considering Figs 1.4-3–5 we can write for potential due volume, surface and line charge distributions as

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{r}\,')\,dV}{|\vec{r}-\vec{r}\,'|} \quad \text{(Volume charge distribution)}$$

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_S \frac{\sigma(\vec{r}\,')\,dS}{|\vec{r}-\vec{r}\,'|} \quad \text{(Surface charge distribution)}$$

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_l \frac{\lambda(\vec{r}\,')\,dl}{|\vec{r}-\vec{r}\,'|} \quad \text{(Line charge distribution)}$$

Equipotential surfaces

An equipotential surface is the locus of points having the same potential. In case of a point charge q located at the origin, the potential at a distance r from the origin is given by

$$\phi(\bar{r}) = \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{r}.$$

Obviously a sphere of radius r with the centre at q will be an equipotential surface. In this case, all concentric spheres with the centre at q will be equipotential surfaces (Fig 1.5-1). In general the shape of the equipotential surface depends on the charge configuration. But whatever may be its shape the relation $\vec{E} = -\vec{\nabla}\phi$ indicates that the *electric field $\vec{E}$ is always perpendicular to the equipotential surface*, $\phi =$ constant, at every point.

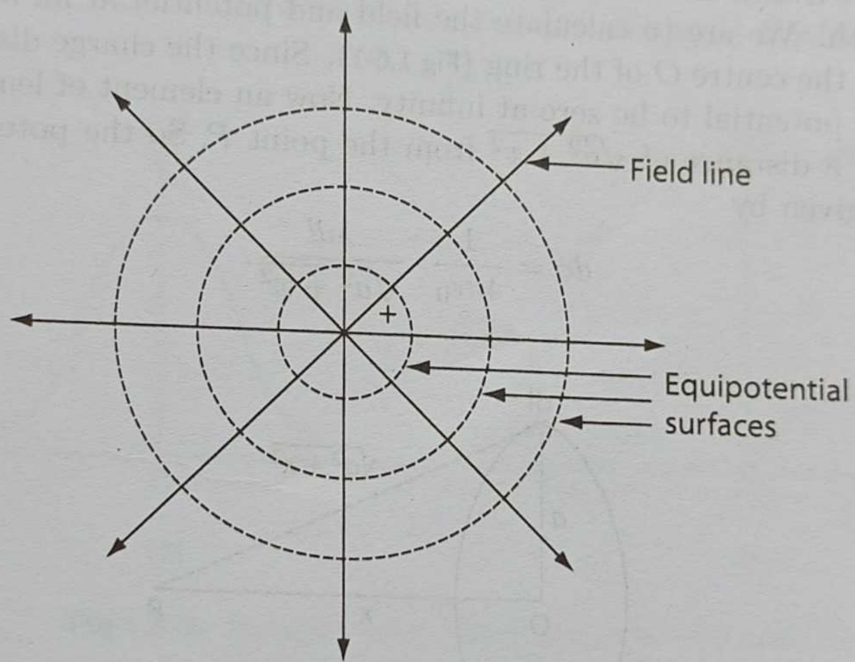


Fig 1.5-1: Field lines and equipotential surfaces for a point charge.

Advantage of the potential concept

Electric field $\vec{E}$ is a vector quantity. Its direct calculation very often becomes tedious and cumbersome. On the other hand, potential ϕ is a scalar quantity. In many cases its calculation is found to be easier. The potential concept reduces a vector problem down to a scalar one. So in practice it is often preferable to determine $\vec{E}$ by first calculating ϕ and then using the relation $\vec{E} = -\vec{\nabla}\phi$, instead of determining $\vec{E}$ directly.

Units of electric field and potential

The SI unit of potential is volt (V). The potential at a point is said to be 1 V if one joule (J) of work is to be done in taking one coulomb (C) of charge from infinity up to that point. Thus one volt equals one joule per coulomb.

By definition $\vec{E}$ is the force per unit charge. Therefore, unit of $\vec{E}$ may be newton per coulomb (N/C). Its most common SI unit is volt per metre ($V.m^{-1}$).

1.6 Calculations of Potential and Field —Some Common Examples

1. A uniformly charged ring

Suppose we have a thin circular ring of radius a having a uniform line charge density of λ per unit length. We are to calculate the field and potential at an axial point P at a distance x from the centre O of the ring (Fig 1.6-1). Since the charge distribution is finite we can take the potential to be zero at infinity. Now an element of length dl containing λdl charge is at a distance of $\sqrt{a^2+x^2}$ from the point P. So the potential at P due to this element is given by

$$d\phi = \frac{1}{4\pi\epsilon_0} \cdot \frac{\lambda dl}{\sqrt{a^2+x^2}}.$$

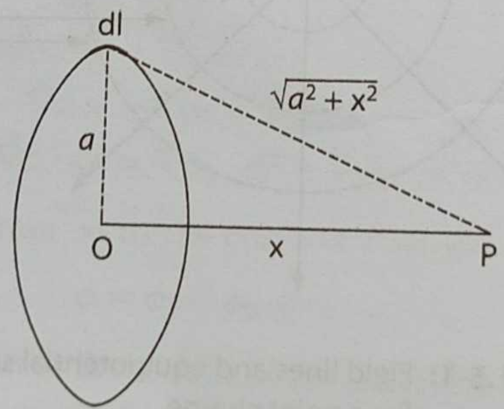


Fig 1.6-1

Since all the length elements are equidistant from P, total potential at P would be given by

$$\phi = \frac{1}{4\pi\epsilon_0} \int \frac{\lambda dl}{\sqrt{a^2+x^2}} = \frac{1}{4\pi\epsilon_0} \cdot \frac{Q}{\sqrt{a^2+x^2}}, \tag{1.6-1}$$

where $Q = \int \lambda dl$ is the total charge on the ring.

Because of perfect symmetry about the axis the electric field at P is directed along the axis and is given by

$$\vec{E} = -\vec{\nabla}\phi = -\hat{x}\frac{d\phi}{dx} = \frac{1}{4\pi\epsilon_0} \cdot \frac{Qx}{(a^2 + x^2)^{3/2}}\hat{x}, \tag{1.6-2}$$

where $\hat{x}$ is a unit vector along OP.

Fig 1.6-2 shows the variation of electric potential $\phi(x)$ and field $E(x)$ with distance x along the axis of the ring. The field is maximum at a distance x for which $\frac{dE}{dx} = 0$. This gives $x = \pm a/\sqrt{2}$.

Note that if the point P is far away from the ring then $x \gg a$ and we can write approximately,

$$\vec{E} \approx \frac{1}{4\pi\epsilon_0} \cdot \frac{Q}{x^2}\hat{x}.$$

In this case the charged ring behaves like a point charge Q placed at the centre of the ring.

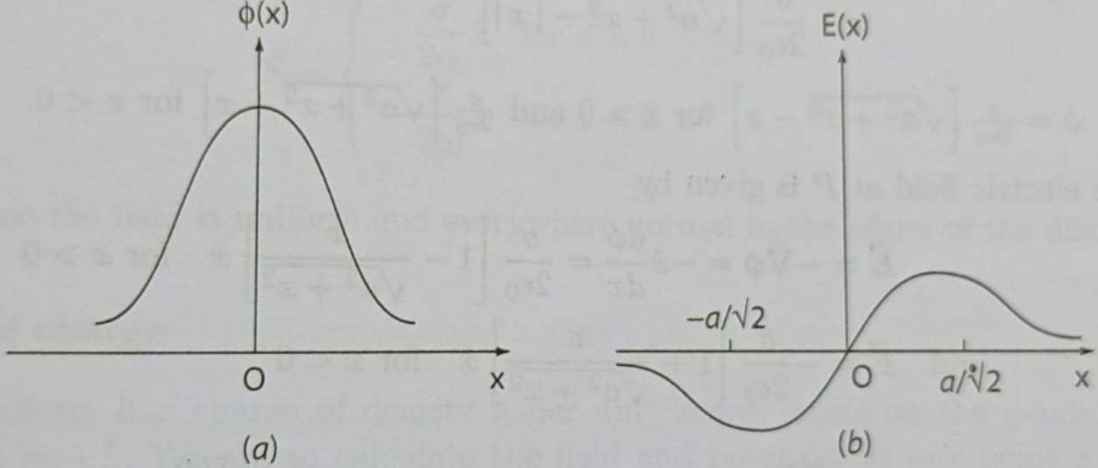


Fig 1.6-2: Variation of **(a)** potential and **(b)** field with distance on the axis of a charged ring.

2. A uniformly charged disc

Suppose we have a thin circular disc of radius a having a uniform surface charge density of σ on it. We are to calculate the field and potential at an axial point P at a distance x from the centre O of the disc (Fig 1.6-3). Let us consider a thin ring of radius r and thickness dr. The charge on this ring is $\sigma.2\pi r\,dr$. Each element of this ring is almost equidistant from P and hence the potential at P due to this ring is given by

$$d\phi = \frac{1}{4\pi\epsilon_0} \cdot \frac{\sigma \cdot 2\pi r\,dr}{\sqrt{r^2 + x^2}}.$$

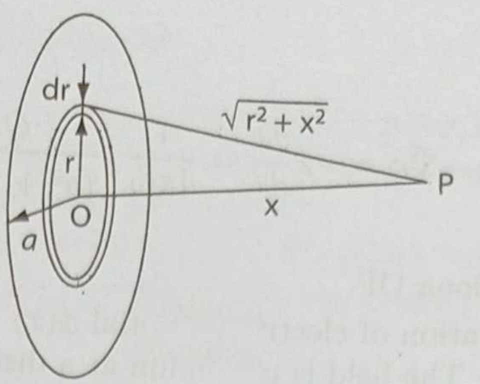


Fig 1.6-3

Therefore, total potential at P due to the whole disc is obtained by summing over all rings from $r = 0$ to $r = a$,

$$\phi = \frac{\sigma}{2\epsilon_0}\int_0^a \frac{rdr}{\sqrt{r^2+x^2}}$$

$$= \frac{\sigma}{2\epsilon_0}\int_x^{\sqrt{a^2+x^2}} \frac{ZdZ}{Z} \quad [\text{Putting } r^2+x^2 = Z^2]$$

$$= \frac{\sigma}{2\epsilon_0}\left[\sqrt{a^2+x^2} - |x|\right] \tag{1.6-3}$$

$\therefore \quad \phi = \frac{\sigma}{2\epsilon_0}\left[\sqrt{a^2+x^2} - x\right]$ for $x > 0$ and $\frac{\sigma}{2\epsilon_0}\left[\sqrt{a^2+x^2} + x\right]$ for $x < 0$.

The electric field at P is given by

$$\vec{E} = -\vec{\nabla}\phi = -\hat{x}\frac{d\phi}{dx} = \frac{\sigma}{2\epsilon_0}\left[1 - \frac{x}{\sqrt{a^2+x^2}}\right]\hat{x} \quad \text{for } x > 0 \tag{1.6-4}$$

$$\text{and} \quad \vec{E} = -\frac{\sigma}{2\epsilon_0}\left[1 + \frac{x}{\sqrt{a^2+x^2}}\right]\hat{x} \quad \text{for } x < 0 \tag{1.6-5}$$

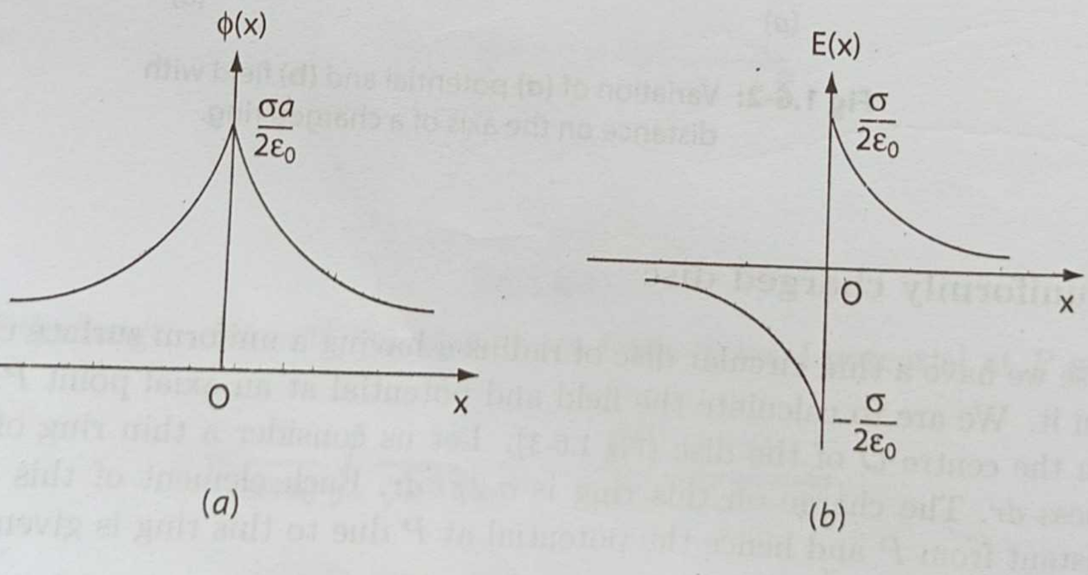


Fig 1.6-4: Variation of **(*a*)** potential and **(b)** field on the axis of a uniformly charged disc.

Fig 1.6-4 shows the variation of electric potential $\phi(x)$ and field $E(x)$ with the distance x. Obviously at the surface $x = 0$, $\phi(x)$ is continuous whereas $E(x)$ is discontinuous. In fact this is true for any surface containing charges.

If the point P is far away from the disc then $|x| \gg a$ and using binomial theorem we can write approximately,

$$\sqrt{a^2 + x^2} = x\left(1 + \frac{a^2}{x^2}\right)^{1/2} \approx x\left(1 + \frac{a^2}{2x^2}\right) = x + \frac{a^2}{2x}.$$

Substituting this in Eq. (1.6-3) we get

$$\phi(x) = \frac{\sigma}{2\epsilon_0} \cdot \frac{a^2}{2x} = \frac{1}{4\pi\epsilon_0} \cdot \frac{Q}{x},$$

where $Q = \pi a^2.\sigma$ is the total charge on the disc. This shows that for points far away from the disc it behaves like a point charge Q placed at the centre of the disc.

If the disc is of infinite extent $(a \to \infty)$ then

$$\vec{E} = \begin{cases} \dfrac{\sigma}{2\epsilon_0}\hat{x} & \text{for } x > 0 \\ -\dfrac{\sigma}{2\epsilon_0}\hat{x} & \text{for } x < 0 \end{cases}$$

In this case the field is uniform and everywhere normal to the plane of the disc.

3. A line of charge

Suppose a uniform line charge of density λ per unit length exists on the y-axis from $y = -L$ to $y = +L$. We are to calculate the field and potential at any point P at a distance x from the line of charge and lying on the perpendicular bisector of the line (Fig 1.6-5). The electric field $d\vec{E}$ at P due to an element dy is of magnitude

$$\left|d\vec{E}\right| = \frac{1}{4\pi\epsilon_0} \cdot \frac{\lambda dy}{R^2}$$

and directed along $\overrightarrow{PA}$. It can be broken into two components $\left|d\vec{E}\right| \cos\theta$ along $\overrightarrow{OP}$ and $\left|d\vec{E}\right| \sin\theta$ perpendicular to $\overrightarrow{OP}$. Now, if we consider another element of line charge placed symmetrically about O it becomes clear that the perpendicular components cancel out and only the horizontal components contribute. Therefore, total field at P due to the whole line of charge would be

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \cdot \int \frac{\lambda\, dy}{R^2} \cos\theta \cdot \hat{x}.$$

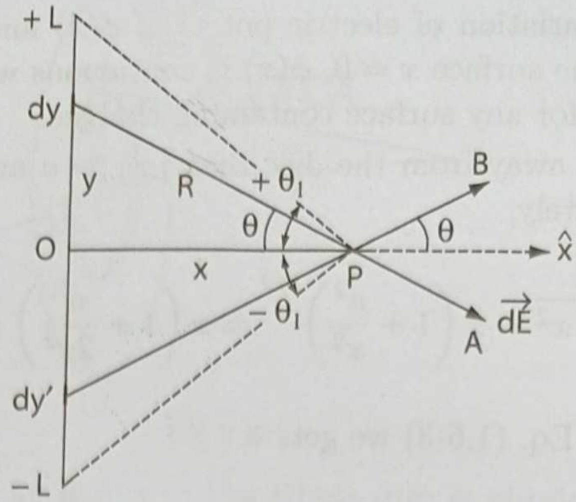


Fig 1.6-5

Now from Fig 1.6-5,

$$y = x\tan\theta, \quad dy = x\sec^2\theta\, d\theta \quad \text{and} \quad x/R = \cos\theta \quad \text{or} \quad R = x\sec\theta.$$

Substituting all these we get,

$$\vec{E} = \frac{\hat{x}\lambda}{4\pi\epsilon_0}\int_{-\theta_1}^{+\theta_1}\frac{\cos\theta\, d\theta}{x} = \frac{\hat{x}\lambda}{4\pi\epsilon_0 x}\cdot 2\sin\theta_1 = \frac{\hat{x}}{4\pi\epsilon_0}\cdot\frac{2\lambda L}{x\sqrt{x^2+L^2}}. \tag{1.6-6}$$

If the point P is far away from the line charge then $x \gg L$ and we get approximately,

$$\vec{E} = \frac{\hat{x}}{4\pi\epsilon_0}\cdot\frac{2\lambda L}{x^2} = \frac{\hat{x}}{4\pi\epsilon_0}\cdot\frac{Q}{x^2},$$

where $Q = 2\lambda L$ is the total line charge. In this case the line charge behaves like a point charge Q placed at the mid-point of the line.

For a line charge of infinite extent or for points very close to the line charge $L \gg x$ or $\theta_1 = \pi/2$ and we can write

$$\vec{E} \approx \frac{\vec{x}}{4\pi\epsilon_0}\cdot\frac{2\lambda}{x} = \frac{\vec{x}\lambda}{2\pi\epsilon_0 x}. \tag{1.6-7}$$

Potential of an infinite line charge

So far in calculating the potential due to a charge distribution we have assume the potential to be zero at infinity. This is allowed for charge distributions located within finite region of space. In this example charge distribution extends to infinity and we cannot take the potential to be zero at infinity. If we try to do so the integral determining the potential diverges and we get infinite result. The work done in taking a unit charge from some reference point x_0 to the point x is given by

$$\phi(x) - \phi(x_0) = -\int_{x_0}^{x} E(x)\, dx = -\int_{x_0}^{x}\frac{\lambda}{2\pi\epsilon_0 x}dx = -\frac{\lambda}{2\pi\epsilon_0}\ln x + \frac{\lambda}{2\pi\epsilon_0}\ln x_0. \tag{1.6-8}$$

This shows that the electric potential due to an infinite line charge can be taken as

$$\phi(x) = -\frac{\lambda}{2\pi\epsilon_0}\ln x + constant. \tag{1.6-9}$$

Note that the constant $\frac{\lambda}{2\pi\epsilon_0}\ln x_0$ in (1.6-8) has no effect when we take $-\vec{\nabla}\phi$ to calculate $\vec{E}$:

$$\vec{E} = -\vec{\nabla}\phi = -\vec{x}\frac{d\phi}{dx} = \frac{\vec{x}\lambda}{2\pi\epsilon_0 x}.$$

1.7 Flux of Electric Field and Gauss's Law

The electric field due to any given charge distribution can, in principle, be determined by using Coulomb's law and the superposition principle. The process is very often found to be tedious. The Gauss's law, which will be discussed now provides a very powerful method of solution of some symmetric electrostatic problems. It is not an independent law but can be derived from Coulomb's law. However, in many cases Gauss's law is found to be more powerful than Coulomb's law. Before introducing this law let us explain two useful ideas — *flux of electric field* and *solid angle*.

Flux of electric field

Consider a surface element $d\vec{S} = \hat{n}\,dS$ in an electric field $\vec{E}$ (Fig 1.7-1), where $\hat{n}$ is the outward unit vector normal to the surface element. The quantity $d\Phi = \vec{E}\cdot d\vec{S} = E\cos\theta dS$ is called the *flux* of $\vec{E}$ through $d\vec{S}$. If we consider that q/ϵ_0 number of lines of force emanate isotropically from a point charge q then the idea of electric flux becomes meaningful. If we use this picture of electric field then $d\Phi = \vec{E}\cdot d\vec{S}$ becomes equal to the number of lines of force passing through the area $d\vec{S}$. The flux of $\vec{E}$ over any arbitrary surface S is given by the integral

$$\Phi = \int d\Phi = \int_S \vec{E}\cdot d\vec{S}$$

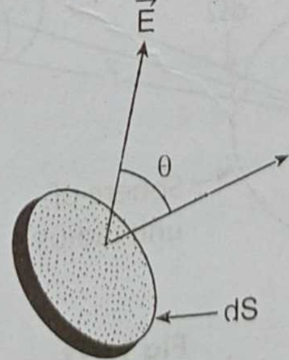


Fig 1.7-1

Solid angle

The concept of solid angle arises as the three dimensional extension of the idea of two dimensional planar angle. The plane angle $d\theta$ subtended by an arc dl of a circle of radius r, at the centre of the circle is defined by

$$d\theta = \frac{dl}{r}.$$

Similarly in three dimension the solid angle $d\Omega$ subtended by a surface element dS of a sphere of radius r, at the centre of the sphere is defined by

$$d\Omega = \frac{dS}{r^2}.$$

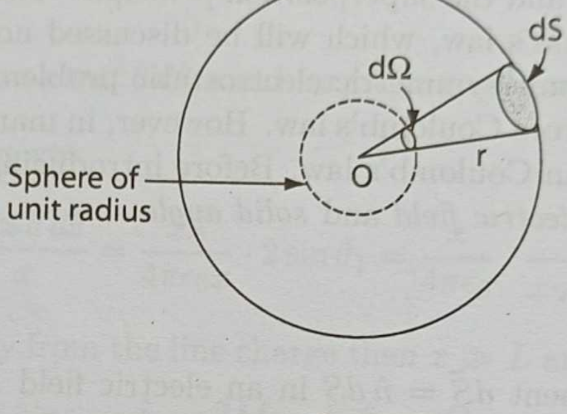


Fig 1.7-2

The solid angle is a dimensionless quantity. Its unit is steradian (sr). If the peripheral points of the surface element dS are connected to O by straight lines, a cone is generated whose base is dS and apex O (Fig 1.7-2). The area that this constructed cone intercepts from the surface of a sphere of unit radius drawn about O is numerically equal to the solid angle $d\Omega$.

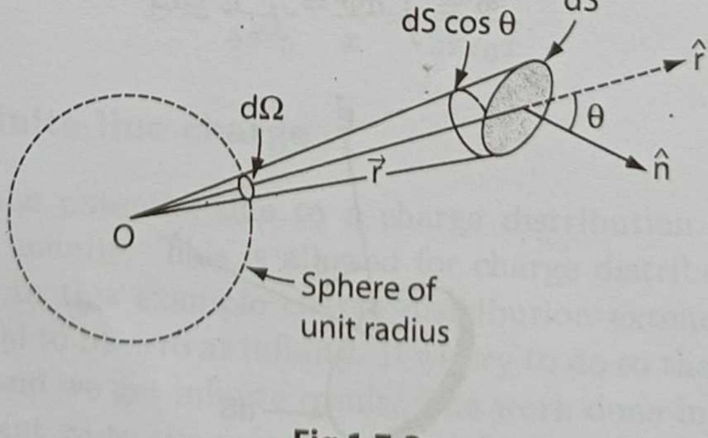


Fig 1.7-3

Now we consider an arbitrary surface element $d\vec{S} = \hat{n}\, dS$ whose position vector with respect a point O is $\vec{r}$ (Fig 1.7-3). If the vector $\vec{r}$ makes an angle θ with $\hat{n}$, the unit vector normal to dS then the projection of the surface element on a plane normal to $\vec{r}$ would be $dS\cos\theta = \hat{r}\cdot\hat{n}\,dS$. In this case the solid angle subtended by $d\vec{S}$ at O would be

$$d\Omega = \frac{dS\cos\theta}{r^2} = \frac{\hat{r}\cdot\hat{n}\,dS}{r^2}. \tag{1.7-1}$$

Remember that total solid angle subtended by a closed surface at any internal point is 4π but at any external point is zero.

Gauss's law

Gauss showed that an important relationship exists between the total electric flux over a closed surface and the total charge enclosed by the surface. This relationship is known as *Gauss's law*. Using SI units the law may be stated as follows:

In an arbitrary electrostatic field (in vacuum) the total electric flux over any closed surface is equal to $1/\epsilon_0$ times the total charge enclosed by the surface, where ϵ_0 is the free space permittivity. For a closed surface S enclosing N number of point charges q_1, $q_2, \ldots, q_N$ the law may be expressed mathematically as

$$\oint_S \vec{E}\cdot d\vec{S} = \frac{1}{\epsilon_0}\times\sum_{i=1}^{N} q_i. \tag{1.7-2}$$

For a continuous distribution of charge within a volume V enclosed by S and characterised by a volume charge density ρ the law may be written as

$$\oint_S \vec{E}\cdot d\vec{S} = \frac{1}{\epsilon_0}\int_V \rho\,dV. \tag{1.7-3}$$

This is known as the *integral form of Gauss's law*. The closed surface on which the Gauss's law is applied is called a *Gaussian surface*.

Proof of Gauss's law

Suppose we consider a point charge q located at the point O and draw a close surface S enclosing q as shown in Fig 1.7-4. Let $d\vec{S} = \hat{n}\,dS$ be a surface element on this surface and the position vector of any point P on this element with respect to O be $\vec{r}$. The electric field $\vec{E}$ at P due to q, i.e., the force on unit positive charge placed at P is given by Coulomb's law as,

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}.$$

The flux of $\vec{E}$ through $d\vec{S}$ is $\vec{E}\cdot d\vec{S} = \vec{E}\cdot\hat{n}\,dS$.

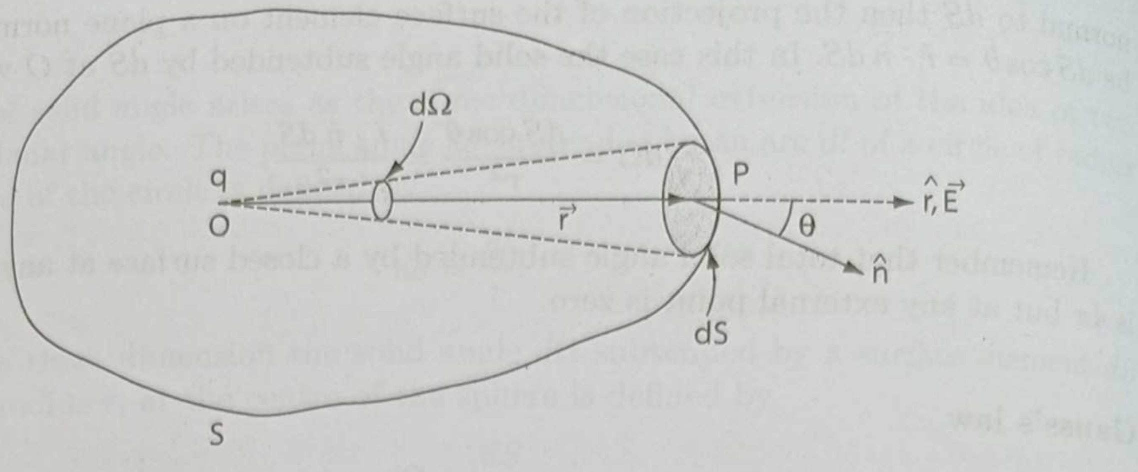


Fig 1.7-4

Therefore, total electric flux over S is

$$\oint_S \vec{E}\cdot\hat{n}\,dS = \frac{q}{4\pi\epsilon_0}\oint_S \frac{\hat{r}\cdot\hat{n}\,dS}{r^2}.$$

Now the quantity $\dfrac{\hat{r}\cdot\hat{n}dS}{r^2} = \dfrac{dS\cos\theta}{r^2} = d\Omega$ is the solid angle subtended by dS at q. Therefore,

$$\oint \vec{E}\cdot\hat{n}\,dS = \frac{q}{4\pi\epsilon_0}\cdot\oint d\Omega = \frac{q}{4\pi\epsilon_0}\times 4\pi = \frac{q}{\epsilon_0} \tag{1.7-4}$$

[$\because$ Total solid angle subtended by S at O is 4π]

Charges may lie outside the surface S. Consider a charge q at a point O outside a closed surface S as shown in Fig 1.7-5. If an elementary cone of solid angle $d\Omega$ be drawn from q, it would cut the surface twice in areas dS_1 and dS_2 at P_1 and P_2.

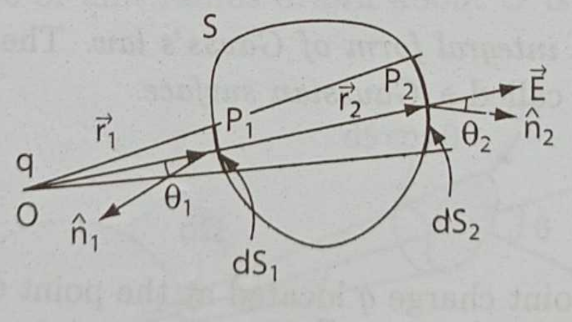


Fig 1.7-5

The electric field lines enter the close surface through dS_1 and leave it through dS_2. As the solid angles subtended by dS_1 and dS_2 at O are the same we can say that the electric flux through dS_1 and dS_2 will be equal and opposite and hence, they contribute nothing to the total electric flux over S. This is true for all cones drawn from O to cover the whole surface S. Thus the contribution to the total electric flux over S will be zero when charges lie outside S. If there are a number of charges $q_1, q_2, \ldots$ inside S then by the principle of superposition total field

$$\vec{E} = \sum_i \vec{E}_i,$$

where $\vec{E}_i$ is the field due to i-th charge q_i. Then Eq. (1.7-4) takes the form

$$\oint_S \vec{E} \cdot \hat{n}\, dS = \frac{1}{\epsilon_0} \sum_i q_i. \tag{1.7-5}$$

As the charges lying outside S do not contribute to the total electric flux over S the electric field $\vec{E}$ in Eq. (1.7-5) may be considered as the total field due to all charges lying within as well as outside S. The result (1.7-5) can also be generalised to the case of a continuous distribution of charges within S. If S encloses a volume V and ρ is the charge density within V then Eq. (1.7-5) changes to

$$\oint_S \vec{E} \cdot \hat{n}\, dS = \frac{1}{\epsilon_0} \int_V \rho\, dV. \tag{1.7-6}$$

Differential form of Gauss's law

Using divergence theorem left-hand side of the Eq. (1.7-6) can be changed into a volume integral,

$$\int_V \vec{\nabla} \cdot \vec{E}\, dV = \frac{1}{\epsilon_0} \int_V \rho\, dV.$$

Since this is true for any volume V, the integrands must be equal, i.e.,

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \rho. \tag{1.7-7}$$

This is the *differential form of Gauss's law*. It is a basic equation in electrostatics. It relates electric field at a point with the charge density at that point.

Since $\vec{E} = -\vec{\nabla}\phi$, Eq. (1.7-7) may be rewritten as

$$\vec{\nabla} \cdot \left(-\vec{\nabla}\phi\right) = \frac{1}{\epsilon_0}\rho \quad \text{or} \quad \nabla^2\phi = -\frac{\rho}{\epsilon_0}. \tag{1.7-8}$$

This is known as *Poisson's equation*. When $\rho = 0$, it reduces to

$$\nabla^2\phi = 0, \tag{1.7-9}$$

which is known as *Laplace's equation*.

The charge $\sigma \Delta S$ on ΔS cannot exert a force on itself but this charge $\sigma \Delta S$ experiences a force due to field $\vec{E}_2$ produced by other charges. Therefore, force on the charges contained in ΔS is

$$\vec{F} = \sigma \Delta S \vec{E}_2 = \frac{\sigma^2 \Delta S}{2\epsilon_0} \hat{n}.$$

So the force on unit area or electrostatic pressure is

$$\vec{P} = \frac{\sigma^2}{2\epsilon_0} \hat{n}.$$

Obviously the pressure acts in the outward direction irrespective of the sign of σ. The pressure can also be expressed in terms of the field given by Eq. (2.2-2) at the conductor surface,

$$\vec{P} = \tfrac{1}{2}\epsilon_0 E^2 \hat{n}.$$

2.3 Electric Dipole

Two equal and opposite charges separated by a very small distance are said to constitute an *electric dipole.* The molecules of a dielectric medium placed in an electrostatic field behave like an electric dipole. So it is important to study the electrostatics of electric dipole before considering the electrostatics in dielectric.

For example, in a H_2O molecule the positive and negative charge centres do not coincide but are separated from each other by a very small distance. Hence, a H_2O molecule can be considered as an electric dipole. We define *dipole moment* $\vec{p}$ of a dipole constituted by two charges $+q$ and $-q$ separated by a distance l as a vector quantity of magnitude ql and directed from $-q$ to $+q$. It has the unit of coulomb-metre. The value of dipole moment for a molecule is usually very small. For this a smaller unit called the *debye unit* is used. 1 debye unit $= 1D = 3.336 \times 10^{-30}$ C.m.

Potential and field due to an electric dipole

Suppose that we have a dipole consisting of a charge $-q$ at the point $\vec{r}'$ and a charge $+q$ at $\vec{r}' + \vec{l}$ as shown in Fig 2.3-1. The electric potential at any arbitrary point $\vec{r}$ is given by

$$\begin{aligned} \phi(\vec{r}) &= \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{\left|\vec{r} - \vec{r}' - \vec{l}\right|} - \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{|\vec{r} - \vec{r}'|} \\ &= \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{\left|\vec{R} - \vec{l}\right|} - \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{\left|\vec{R}\right|}. \end{aligned} \quad (2.3\text{-}1)$$

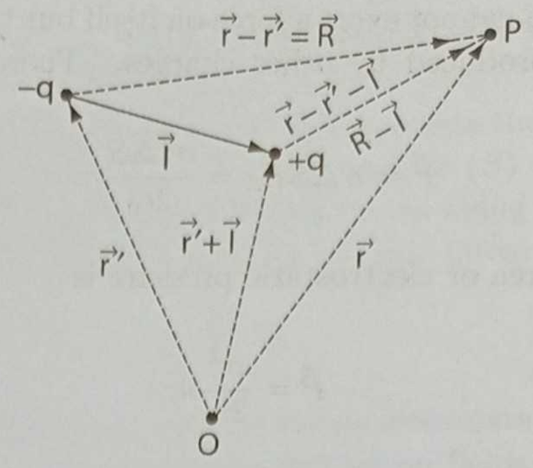


Fig 2.3-1

If the distance $\left|\vec{R}\right| = R$ is large compared with the dipole length l then we can make the following expansion:

$$\frac{1}{|\vec{R}-\vec{l}|} = \frac{1}{\left|\left(\vec{R}-\vec{l}\right)\cdot\left(\vec{R}-\vec{l}\right)\right|^{1/2}}$$

$$= \frac{1}{\left|R^2 - 2\vec{R}\cdot\vec{l} + l^2\right|^{1/2}}$$

$$= \frac{1}{R}\left[1 + \frac{\vec{R}\cdot\vec{l}}{R^2} + \cdots\right]. \tag{2.3-2}$$

$\therefore$ From Eq. (2.3-1) we get

$$\phi(\vec{r}) = \frac{q}{4\pi\epsilon_0}\cdot\frac{\vec{R}\cdot\vec{l}}{R^3}, \tag{2.3-3}$$

where we have kept only terms linear in l. This expression is valid only for $|\vec{R}| \gg l$. In many cases the concept of *point* or *pure dipole* becomes useful. Here $l \to 0$ and $q \to \infty$ in such a away that the product ql remains constant. In this case, Eq. (2.3-3) becomes exact. Now as $q\vec{l} = \vec{p}$ is the dipole moment Eq. (2.3-3) can be written as

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\vec{R}}{R^3}. \tag{2.3-4}$$

$$\text{As}\quad \vec{\nabla}\frac{1}{R} = -\frac{\vec{R}}{R^3} \quad\text{and}\quad \vec{\nabla}'\frac{1}{R} = \frac{\vec{R}}{R^3}.$$

Equation (2.3-4) can also be written as

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\vec{p}\cdot\vec{\nabla}'\frac{1}{R} \quad\text{or}\quad \phi(\vec{r}) = -\frac{1}{4\pi\epsilon_0}\vec{p}\cdot\vec{\nabla}\frac{1}{R} = -\vec{p}\cdot\vec{\nabla}\phi_0, \tag{2.3-5}$$

where $\phi_0 = 1/4\pi\epsilon_0 R$ is the potential of a unit point charge.

The electric field at $P(\vec{r})$ can be obtained as

$$\vec{E}(\vec{r}) = -\vec{\nabla}\phi(\vec{r}).$$

Using Eq. (2.3-4) we get

$$\vec{E}(\vec{r}) = -\frac{1}{4\pi\epsilon_0}\left[\frac{1}{R^3}\vec{\nabla}\left(\vec{p}\cdot\vec{R}\right) + \vec{p}\cdot\vec{R}\vec{\nabla}\frac{1}{R^3}\right].$$

Now from vector analysis

$$\vec{\nabla}\left(\vec{p}\cdot\vec{R}\right) = \vec{p} \quad \text{and} \quad \vec{\nabla}\frac{1}{R^3} = -\frac{3\vec{R}}{R^5}$$

$$\therefore \quad \vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{3\left(\vec{p}\cdot\vec{R}\right)\vec{R}}{R^5} - \frac{\vec{p}}{R^3}\right]. \qquad (2.3\text{-}6)$$

The electric field can also be calculated directly by using Coulomb's law.

For convenience let us now choose the charge $-q$ at the origin $\left(\text{i.e., } \vec{r}\,' = 0, \text{ and } \vec{R} = \vec{r}\right)$ and the charges lie along z-axis (Fig 2.3-2). Let θ be the angle between the axis of the dipole and the radius vector to the point of observation P. In this case, Eq. (2.3-4) takes the form

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3} = \frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{r^2}. \qquad (2.3\text{-}7)$$

Note that the dipolar potential varies as $1/r^2$, whereas the potential due an isolated charge (monopole) varies as $1/r$.

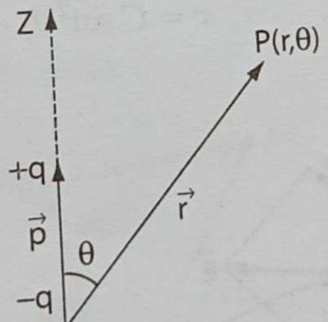


Fig 2.3-2

Very often it becomes convenient to express electric field at P in terms of spherical polar coordinates (r, θ, φ). Using Eq. (2.3-7) we get

$$\left[\because\ \vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right]$$

$$E_r = -\frac{\partial\phi}{\partial r} = \frac{1}{4\pi\epsilon_0}\frac{2p\cos\theta}{r^3}$$

$$E_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{1}{4\pi\epsilon_0}\cdot\frac{p\sin\theta}{r^3}$$

$$E_\varphi = -\frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi} = 0. \quad (2.3\text{-}8)$$

Thus,

$$\vec{E} = \frac{p}{4\pi\epsilon_0 r^3}\left(2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right). \quad (2.3\text{-}9)$$

The magnitude of $\vec{E}$ is

$$\left|\vec{E}\right| = \frac{p}{4\pi\epsilon_0 r^3}\sqrt{4\cos^2\theta + \sin^2\theta} = \frac{p}{4\pi\epsilon_0 r^3}\sqrt{1 + 3\cos^2\theta}.$$

If β is the angle made by $\vec{E}$ with the radius vector then

$$\tan\beta = \frac{E_\theta}{E_r} = \frac{1}{2}\tan\theta \quad \text{or} \quad \beta = \tan^{-1}\left(\frac{1}{2}\tan\theta\right).$$

Note that the dipole field falls off as $1/r^3$ whereas the field of an isolated charge (monopole) falls off as $1/r^2$.

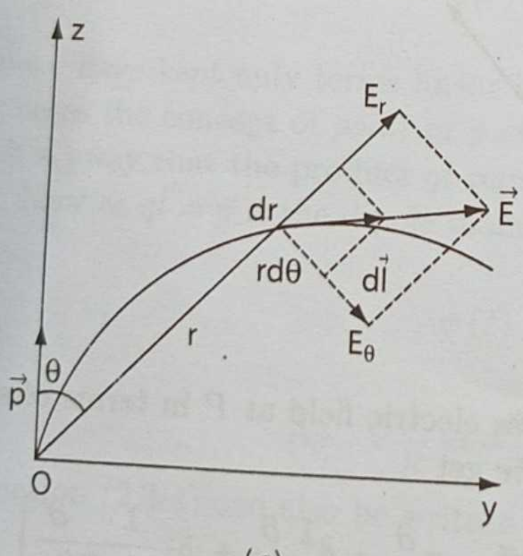
Equipotential surfaces and field lines of a dipole

The equation of lines of force may be obtained from Fig 2.3-3(a) which shows a line element $\vec{dl} = \hat{r}dr + \hat{\theta}rd\theta$ and electric field $\vec{E} = \hat{r}E_r + \hat{\theta}E_\theta$ parallel to each other. Therefore,

$$\frac{rd\theta}{dr} = \frac{E_\theta}{E_r} = \frac{\sin\theta}{2\cos\theta} \quad \text{or} \quad \frac{dr}{r} = \frac{2d(\sin\theta)}{\sin\theta}.$$

Integrating we get,

$$r = C\sin^2\theta.$$



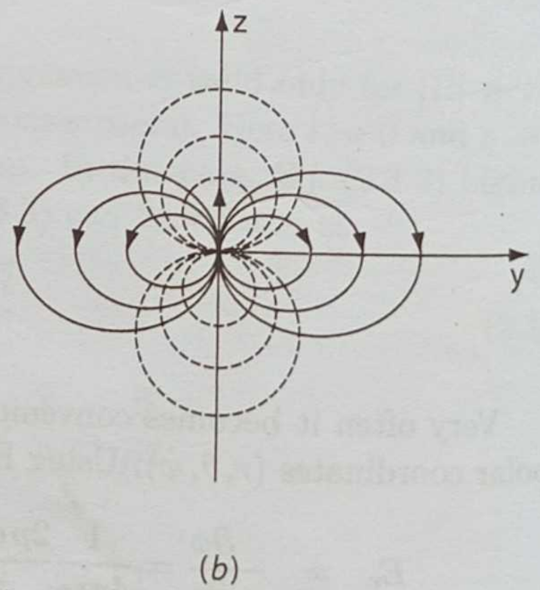


Fig 2.3-3: Equipotential lines (dotted lines) and lines of force (lines with arrows) of the dipole shown in (b).

This equation gives a family of liens of force, where the constant C is different for different lines.

Equipotential surfaces are everywhere perpendicular to the lines of force. Traces of equipotential surfaces in the yz-plane can be obtained as follows. The potential at any point (r, θ) is

$$\phi(r, \theta) = \frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{r^2}.$$

Therefore, for an equipotential line,

$$r(\theta) = A\sqrt{\cos\theta},$$

where $A = \sqrt{p/4\pi\epsilon_0\phi}$ = constant. This equation gives a family of equipotential lines, where A is different for different lines.

Potential energy of a dipole placed in an external electric field

Suppose we have an electric dipole consisting of a charge $-q$ at the point $\vec{r}$ and a charge $+q$ at $\vec{r} + \vec{l}$ as shown in Fig 2.3-4. It is placed in an external electric field $\vec{E}$ described by the potential function $\phi(\vec{r})$. So the potential energy of the dipole is

$$U = -q\phi(\vec{r}) + q\phi\left(\vec{r} + \vec{l}\right). \tag{2.3-10}$$

If $\left|\vec{l}\right| \ll |\vec{r}|$ then $\phi(\vec{r} + \vec{l})$ can be expanded in a Taylor series as

$$\phi(\vec{r} + \vec{l}) = \phi(\vec{r}) + \vec{l} \cdot \vec{\nabla}\phi(\vec{r}), \tag{2.3-11}$$

where we neglect higher order smaller terms.

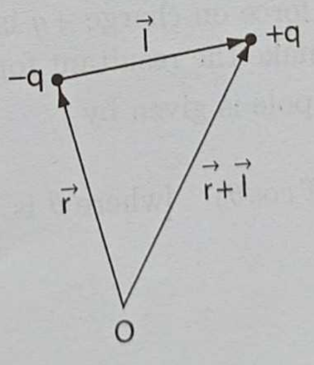


Fig 2.3-4

Using this expansion in Eq. (2.3-10) we get

$$U = q\vec{l} \cdot \vec{\nabla}\phi(\vec{r}) = -\vec{p} \cdot \vec{E}(\vec{r}), \tag{2.3-12}$$

where $\vec{p} = \overrightarrow{ql}$ is the dipole moment and $\vec{E}(\vec{r}) = -\vec{\nabla}\phi(\vec{r})$ is the external electric field.

If the angle between $\vec{p}$ and $\vec{E}$ be θ then potential energy

$$U = -pE\cos\theta. \tag{2.3-13}$$

Note that $\theta = 90°$ is chosen as the position of zero potential energy.

Force and couple on a dipole placed in an electric field

The potential energy W of a dipole of dipole moment $\vec{p}$ placed in an external electric field $\vec{E}$ is

$$U = -\vec{p}\cdot\vec{E}.$$

The force on the dipole due to the electric field is given by[1]

$$\vec{F} = -\vec{\nabla}U = \vec{\nabla}\left(\vec{p}\cdot\vec{E}\right).$$

Now $\vec{p}$ is a constant vector and $\vec{\nabla}\times\vec{E} = 0$. So we can write from the vector identity

$$\vec{\nabla}\left(\vec{A}\cdot\vec{B}\right) = \left(\vec{A}\cdot\vec{\nabla}\right)\vec{B} + \left(\vec{B}\cdot\vec{\nabla}\right)\vec{A} + \vec{A}\times\left(\vec{\nabla}\times\vec{B}\right) + \vec{B}\times\left(\vec{\nabla}\times\vec{A}\right)$$

that

$$\vec{\nabla}\left(\vec{p}\cdot\vec{E}\right) = \left(\vec{p}\cdot\vec{\nabla}\right)\vec{E}.$$

Therefore,

$$\vec{F} = \left(\vec{p}\cdot\vec{\nabla}\right)\vec{E}. \tag{2.3-14}$$

It is a general expression valid for both uniform and nonuniform field.

If the field $\vec{E}$ is uniform then $\vec{F} = 0$ and there will be no translational motion of the dipole. For a uniform field the force on charge $+q$ is $q\vec{E}$ and that on $-q$ is $-q\vec{E}$. These two equal and opposite forces make the resultant force zero.

The torque acting on the dipole is given by

$$\begin{aligned}\Gamma = -\frac{\partial U}{\partial\theta} &= -\frac{\partial}{\partial\theta}(pE\cos\theta) \quad \text{[where } \theta \text{ is the angle between } \vec{p} \text{ and } \vec{E}\text{]}\\ &= pE\sin\theta.\end{aligned}$$

In vector form

$$\vec{\tau} = \vec{p}\times\vec{E}. \tag{2.3-15}$$

---

[1]Work done in an elementary displacement $d\vec{r}$ is

$$\begin{aligned}\vec{F}\cdot d\vec{r} &= -dU \quad \text{(decrease in P.E.)}\\ &= -\vec{\nabla}U\cdot d\vec{r}.\end{aligned}$$

Since $d\vec{r}$ is arbitrary we have $\vec{F} = -\vec{\nabla}U$.

external electric field appears because in these cases no sufficient time is available for neutralisation of polarization charges. The phenomenon of the appearance of electric effect due to thermal expansion of dielectric is known as *pyroelectricity*. The phenomenon of the appearance of electric effect due to mechanical stress on dielectric is known as *piezoelectric effect.*

There is another special class of crystalline dielectric which exhibits spontaneous polarization in the absence of external field below certain temperature and above this temperature it behaves like an ordinary dielectric material with a very large dielectric constant. Such dielectrics are called *ferroelectric* in analogy with the ferromagnetic effect. If a ferroelectric material is subjected to a cycle of polarizing and depolarizing fields the polarization P passes through a loop (hysteresis curve) and it always lags behind the polarizing field (Fig 2.12-1).

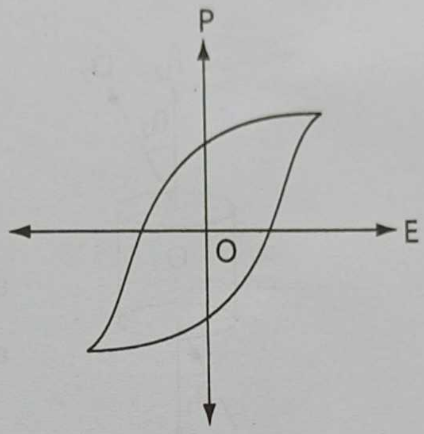


Fig 2.12-1

2.13 The Boundary Conditions on $\vec{E}$ and $\vec{D}$

The boundary conditions on $\vec{E}$ and $\vec{D}$ tell us how these fields change across the boundary surface separating two media.

Let us consider two media, 1 and 2, with S_1S_2 as the surface of separation (Fig 2.13-1). Let there is a free surface density of charges σ on the surface. We construct a small Gaussian pillbox on the surface at O. Its flat surface area ΔS is assumed to be small and its height approaches zero such that the flat faces are arbitrarily close to the boundary. Therefore, from Gauss's law,

$$\int_S \vec{D}\cdot\hat{n}\,dS = \sigma\cdot\Delta S \quad \text{or} \quad \vec{D}_2\cdot\hat{n}_2\Delta S + \vec{D}_1\cdot\hat{n}_1\Delta S = \sigma\Delta S \quad \text{or} \quad \left(\vec{D}_2-\vec{D}_1\right)\cdot\hat{n}=\sigma,$$

where we take $\hat{n}_2 = -\hat{n}_1 = \hat{n}$, unit vector pointing from medium 1 to medium 2 normal to the interface.

$$\therefore \quad \vec{D}_2\cdot\hat{n} - \vec{D}_1\cdot\hat{n} = \sigma \tag{2.13-1}$$

$$\text{or} \quad D_2\cos\theta_2 - D_1\cos\theta_1 = \sigma \tag{2.13-2}$$

Thus, the normal components of $\vec{D}$ are discontinuous across the boundary by the surface charge density. If the two media have permittivities ϵ_1 and ϵ_2 then condition (2.13-1) can be expressed as

$$\epsilon_2 \vec{E}_2 \cdot \hat{n} - \epsilon_1 \vec{E}_1 \cdot \hat{n} = \sigma \quad \text{or} \quad -\epsilon_2 \vec{\nabla}\phi_2 \cdot \hat{n} + \epsilon_1 \vec{\nabla}\phi_1 \cdot \hat{n} = \sigma. \tag{2.13-3}$$

If there is no free surface charge on the boundary then $\vec{D}_2 \cdot \hat{n} = \vec{D}_1 \cdot \hat{n}$, i.e., the normal components of $\vec{D}$ are continuous across the boundary.

If the medium 1 is a conducting medium then $\vec{D}_1 = 0$ and the relevant boundary condition is

$$\vec{D}_2 \cdot \hat{n} = \sigma. \tag{2.13-4}$$

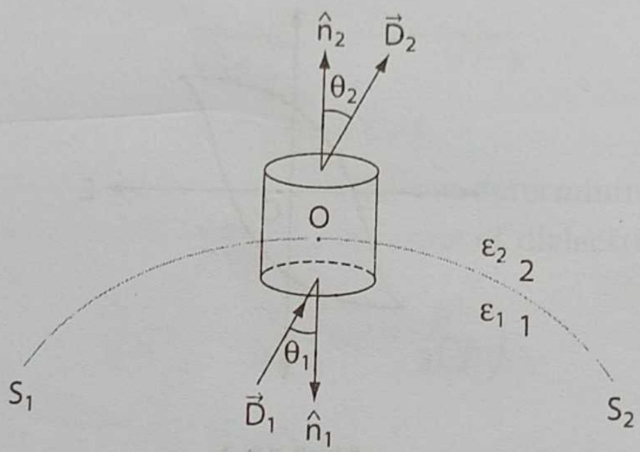


Fig 2.13-1

To find conditions on $\vec{E}$ let us consider a small rectangular path $ABCD$ with sides AB and CD being perpendicular to the surface of separation S_1S_2 and other sides AD and BC being parallel to the surface at O (Fig 2.13-2). Let the sides $AD = BC = \Delta l$ and AB and $CD(= \Delta h)$ negligibly small. Now for any electrostatic field $\vec{E}$

$$\oint \vec{E} \cdot d\vec{l} = 0.$$

Applying this result to the rectangular path $ABCD$ of Fig 2.13-2 we get

$$\vec{E}_2 \cdot \overrightarrow{\Delta l} - \vec{E}_1 \cdot \overrightarrow{\Delta l} = 0$$

$$\text{or} \quad \vec{E}_2 \cdot \hat{\tau} = \vec{E}_1 \cdot \hat{\tau} \tag{2.13-5}$$

$$\text{or} \quad E_2 \sin\theta_2 = E_1 \sin\theta_1, \tag{2.13-6}$$

here $\hat{\tau}$ is a unit vector along the tangent to the interface. Thus, *the tangential component of the electric is continuous across the boundary.*

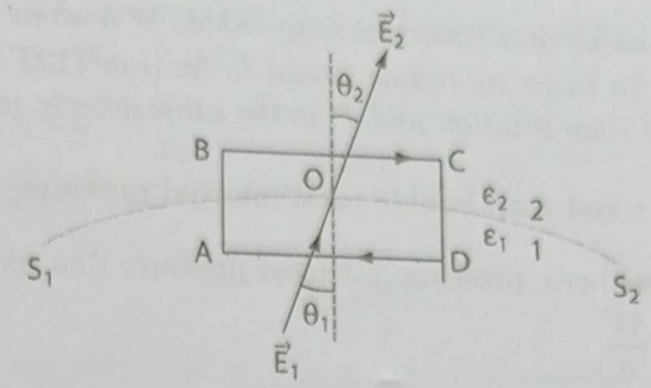


Fig 2.13-2

In terms of potentials condition (2.13-6) takes the form

$$\phi_2 = \phi_1.$$

This means that the potential is continuous across the boundary. This follows from the definition of $\Delta\phi$ as the work done to displace a unit charge between two points separated by a distance $\Delta\vec{r}$. If $\vec{E}$ is finite then as $\Delta\vec{r} \to 0$ across the surface $\vec{E}\cdot\Delta\vec{r} \to 0$, implying that

$$\Delta\phi \to 0 \quad \text{or} \quad \phi_2 = \phi_1.$$

If $\sigma = 0$ then from Eqs. (2.13-2) and (2.13-6) we can write

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\epsilon_1}{\epsilon_2}.$$

This is the law of refraction of the electric field at a boundary free of charge.

SOLVED PROBLEMS

Problem 1. *An electric charge Q is given to a conducting sphere of radius R. Show that the force on a small charge element ΔQ on its surface is $\frac{Q\cdot\Delta Q}{8\pi\epsilon_0 R^2}$.*

Solution : The charge Q gets distributed over the surface of the sphere and surface charge density becomes $\sigma = \frac{Q}{4\pi R^2}$. If ΔF be the force on an element of area ΔS containing $\Delta Q = \sigma\Delta S$ charge then outward mechanical pressure becomes

$$\frac{\Delta F}{\Delta S} = \frac{\sigma^2}{2\epsilon_0}.$$

$$\therefore \quad \Delta F = \frac{\sigma}{2\epsilon_0}\cdot\sigma\Delta S = \frac{\sigma}{2\epsilon_0}\Delta Q = \frac{Q\cdot\Delta Q}{8\pi\epsilon_0 R^2}.$$

Chapter 6

Method of Electrical Image

6.1 Introduction

Various methods are available for the solution of electrostatic problems. Most straight forward method is to solve Laplace's or Poisson's equations with proper boundary conditions. However, the method is not found to be always simple. There are many special problems in electrostatics, which can be solved with greater ease by the method of images, invented by Lord Kelvin. The essence of the method consists in determining the potential due to a given charge distribution by considering a different charge configuration, which satisfies the given boundary conditions and then exploiting the uniqueness theorem. The method is particularly useful for problems consisting of point charges near conductors. To find the potential outside the conductor the induced charges on the conductor are replaced by one or more fictitious point charges of suitable magnitudes at suitable locations such that the equipotential conditions at the conductor surfaces and Laplace's or Poisson's equation is satisfied everywhere in the region outside the conductors. According to uniqueness theorem the potential thus found will be the correct one for the entire region outside the conductors. These fictitious, unreal, imaginary charges are called *image charges* (from the analogy with optical images in mirror reflection). The method is illustrated in this chapter with a few examples.

6.2 Point Charge in Front of an Earthed Conducting Plane

Suppose we have a large grounded conducting plane forming xy-plane and a point charge $+q$ placed on the z-axis at $z = d$ (Fig 6.2-1). We wish to find the potential and field in the region $z \geq 0$. The potential will be partly due to q and partly due to induced charges on the plane. To find the latter contribution we must have a knowledge about the induced charge distribution. In the method of electrical images we try to replace this induced charges by image charges. In this attempt the following conditions must be obeyed:

(*i*) Potential $\phi = 0$ all over the conducting plane ($z = 0$).

(*ii*) ϕ must be zero at infinity.

(*iii*) ϕ satisfies Laplace's equation in the region $z > 0$ except at the point A where the charge q is located.

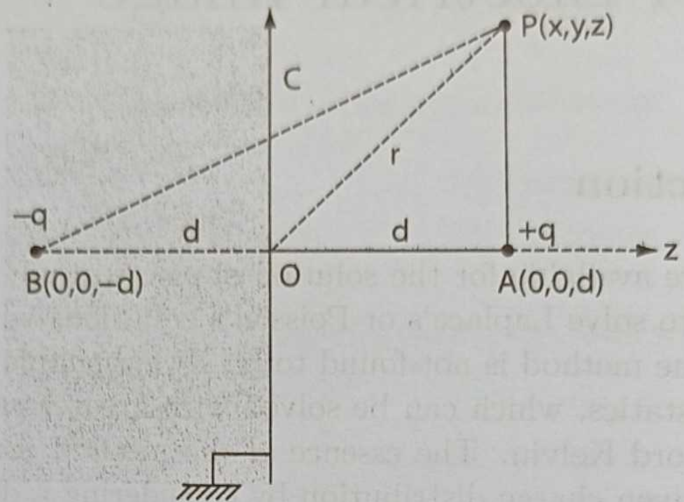


Fig 6.2-1

The symmetry of the problem suggests an image charge $-q$ located at $z = -d$. Let us now see that this choice satisfies all the conditions (*i*)–(*iii*). The potential at any point C on the plane due to q at A and the image charge $-q$ at B is

$$\phi_C = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{AC} - \frac{q}{BC}\right) = 0.$$

The placing of $-q$ at a finite distance does not affect the condition (*ii*) at infinity. As the image charge $-q$ is located in the region $z < 0$, it will not affect the condition (*iii*) in the region $z > 0$. Thus, our choice satisfies all the conditions. According to the uniqueness theorem the solution thus found is unique. So instead of the actual induced charges the image charge can be used to find the potential and field in the region $z > 0$. The actual induced charge distribution can also be found.

Potential and field at any point $P(x, y, z)$

$$\begin{aligned}\phi(x, y, z) &= \frac{1}{4\pi\epsilon_0}\left(\frac{q}{AP} - \frac{q}{BP}\right) \\ &= \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{x^2 + y^2 + (z-d)^2}} - \frac{1}{\sqrt{x^2 + y^2 + (z+d)^2}}\right). \qquad (6.2\text{-}1)\end{aligned}$$

The electric field can be determined by using the relation

$$\vec{E} = -\vec{\nabla}\phi = -\left[\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right]. \tag{6.2-2}$$

Lines of force

The electric lines of force will be the same as that between the charge $+q$ at A and $-q$ at B (Fig 6.2-2). The lines of force on the side $z < 0$ is shown by dotted lines because they do not exist really, the field there is zero. The field lines meet the plane normally.

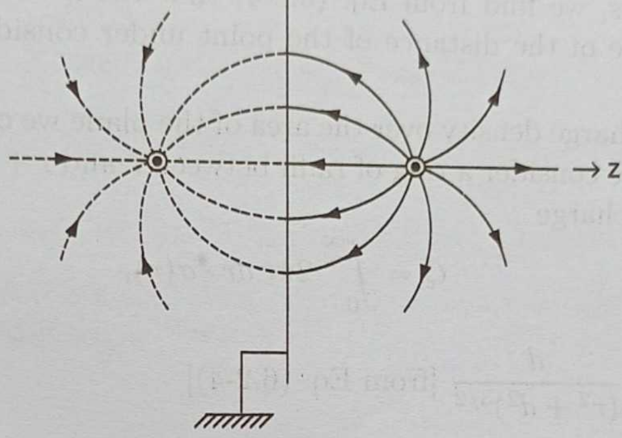


Fig 6.2-2

Force on $+q$

Force on $+q$ due to the conducting plane is given by the force between $+q$ at A and its image charge $-q$ at B. Therefore, the force on $+q$ due to the conducting plane is

$$\vec{F} = -\frac{1}{4\pi\epsilon_0} \cdot \frac{q^2}{(2d)^2}\hat{k}, \tag{6.2-3}$$

where $\hat{k}$ is the unit vector along OA.

The force is attractive in nature. This force is largely responsible for preventing the free electrons from leaving the surface of conductor and is associated with the work function of the conducting material.

Induced charges

The actual induced surface charge density σ can be obtained as follows

$$\sigma = -\epsilon_0 \frac{\partial \phi}{\partial z}\bigg|_{z=0}.$$

From Eq. (6.2-1) we have

$$\frac{\partial \phi}{\partial z} = \frac{q}{4\pi\epsilon_0}\left[\frac{-(z-d)}{\{x^2+y^2+(z-d)^2\}^{3/2}} + \frac{z+d}{\{x^2+y^2+(z+d)^2\}^{3/2}}\right].$$

$$\therefore \quad \sigma(x,y) = -\frac{q}{2\pi}\cdot\frac{d}{(x^2+y^2+d^2)^{3/2}}. \qquad (6.2\text{-}4)$$

The induced charge, as expected, is negative. It has a maximum value at $x = y = 0$ on the plane. The distance of any point $(x, y, 0)$ on the plane from $A(0, 0, d)$ is $\sqrt{x^2 + y^2 + d^2}$. Thus, we find from Eq. (6.2-4) that the induced charge density varies inversely as the cube of the distance of the point under consideration from the charge $+q$ at A.

Integrating the charge density over the area of the plane we can find the total induced charge. To do this we consider a ring of radii between r and $r + dr$, where $r = \sqrt{x^2 + y^2}$. Then total induced charge

$$Q = \int_0^\infty 2\pi r\, dr \cdot \sigma(r),$$

where $\sigma(r) = -\frac{q}{2\pi} \cdot \frac{d}{(r^2+d^2)^{3/2}}$ [from Eq. (6.2-4)]

$$\begin{aligned}\therefore \quad Q &= -qd\int_0^\infty \frac{r\,dr}{(r^2+d^2)^{3/2}} \\ &= -qd\left[-\frac{1}{\sqrt{r^2+d^2}}\right]_0^\infty \\ &= -q. \qquad (6.2\text{-}5)\end{aligned}$$

Thus, the total induced charge, as expected, is $-q$. This implies that all the lines of force ultimately terminate on the plane.

6.3 A Point Charge and a Grounded Conducting Sphere

Consider a point charge q placed at a distance d from the centre of a grounded conducting sphere of radius a (Fig 6.3-1). The potential at any external point P will be partly due to q and partly due to induced charges on the sphere. To solve the problem by the method of image we are to replace the induced charges by suitable image charges. The symmetry of the problem suggests that the image charge must be placed symmetrically about the line of symmetry OA. Let us try a single image charge q' located within the sphere at some point B (with $OB = d'$) on the line OA. In this attempt the following conditions must be obeyed:

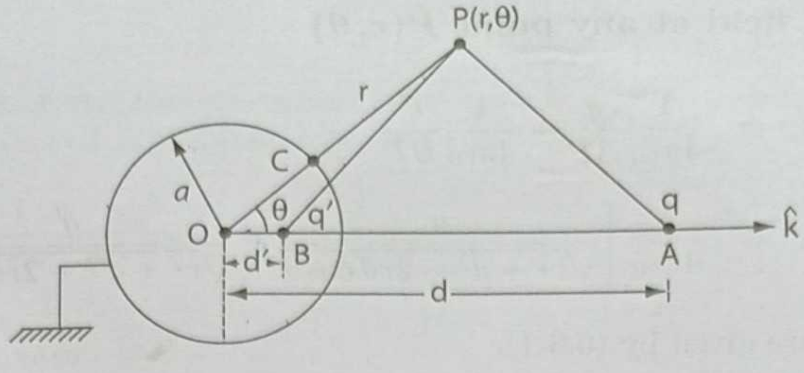


Fig 6.3-1

(*i*) Potential ϕ is zero all over the surface of the sphere.

(*ii*) Potential ϕ must be zero at infinity.

(*iii*) Laplace's equation must be satisfied at all outside points excepting at A.

The choice of image charge q' inside the sphere does not affect the conditions (*ii*) and (*iii*). Now the potential at any point $C(a, \theta)$ on the surface of the sphere must be zero, i.e.,

$$\frac{1}{4\pi\epsilon_0}\left[\frac{q}{AC} + \frac{q'}{BC}\right] = 0$$

$$\text{or} \quad \frac{q}{\sqrt{a^2 + d^2 - 2ad\cos\theta}} + \frac{q'}{\sqrt{a^2 + d'^2 - 2ad'\cos\theta}} = 0$$

$$\text{or} \quad \frac{q/a}{\sqrt{1 + \frac{d^2}{a^2} - 2\frac{d}{a}\cos\theta}} = \frac{-q'/d'}{\sqrt{1 + \frac{a^2}{d'^2} - 2\frac{a}{d'}\cos\theta}}.$$

To make this equality valid for all values of θ we may choose

$$\frac{q}{a} = -\frac{q'}{d'} \quad \text{and} \quad \frac{d}{a} = \frac{a}{d'}.$$

Therefore,

$$q' = -\frac{qa}{d} \quad \text{and} \quad d' = \frac{a^2}{d}. \qquad (6.3\text{-}1)$$

This choice of image charge satisfies all the boundary conditions for our original problem in the region outside the sphere. According to the uniqueness theorem the solution thus obtained is the only correct solution to the problem. So the image charge can now be used instead of the induced charges to find potential, field etc.

Potential and field at any point $P(r, \theta)$

$$\begin{aligned}\phi(r,\theta) &= \frac{1}{4\pi\epsilon_0}\frac{q}{AP} + \frac{1}{4\pi\epsilon_0}\frac{q'}{BP} \\ &= \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{r^2+d^2-2rd\cos\theta}} + \frac{q'}{\sqrt{r^2+d'^2-2rd'\cos\theta}}\right],\end{aligned} \quad (6.3\text{-}2)$$

where q' and d' are given by (6.3-1).

Corresponding field is given by

$$\vec{E} = -\vec{\nabla}\phi = -\left[\hat{r}\frac{\partial\phi}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right].$$

Therefore, the components of $\vec{E}$ are:

$$E_r = -\frac{\partial\phi}{\partial r} = \frac{1}{4\pi\epsilon_0}\left[\frac{q(r-d\cos\theta)}{(r^2+d^2-2rd\cos\theta)^{3/2}} + \frac{q'(r-d'\cos\theta)}{(r^2+d'^2-2rd'\cos\theta)^{3/2}}\right] \quad (6.3\text{-}3)$$

$$E_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{1}{4\pi\epsilon_0}\left[\frac{qd\sin\theta}{(r^2+d^2-2rd\cos\theta)^{3/2}} + \frac{q'd'\sin\theta}{(r^2+d'^2-2rd'\cos\theta)^{3/2}}\right] \quad (6.3\text{-}4)$$

Lines of force

Lines of force will be the same as that between q at A and $q' = -qa/d$ at B (Fig 6.3-2). The lines meet the conducting sphere normally. The lines of force within the sphere are shown by dotted lines because field inside the sphere is zero and they do not exist really.

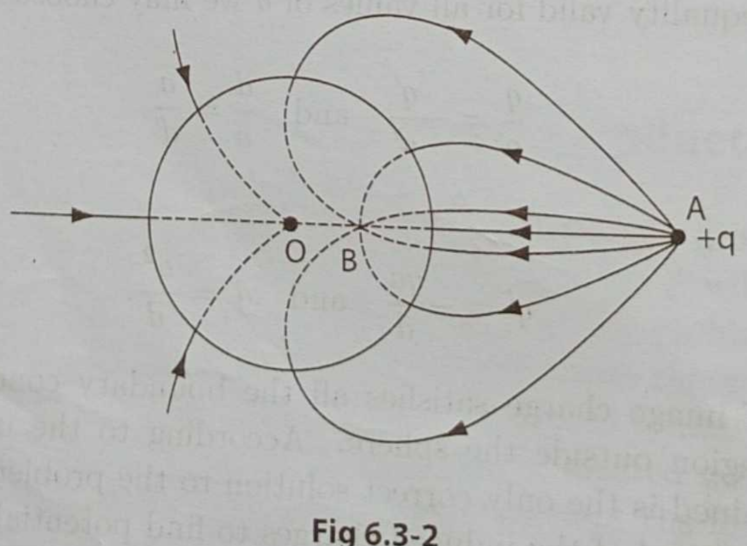


Fig 6.3-2

Force on $+q$

Force between $+q$ at A and the sphere will be given by the force between $+q$ at A and the image charge q' placed at B. Therefore, force on $+q$ due to the sphere would be

$$\begin{aligned} \vec{F} &= \frac{1}{4\pi\epsilon_0}\cdot\frac{qq'}{(BA)^2}\hat{k} \quad \text{where } \hat{k} \text{ is a unit vector along } OA \\ &= -\frac{1}{4\pi\epsilon_0}\cdot\frac{q^2a/d}{(d-a^2/d)^2}\hat{k} \quad [\because\ BA = OA - OB = d - a^2/d] \\ &= -\frac{q^2ad}{4\pi\epsilon_0(d^2-a^2)^2}\hat{k}. \end{aligned} \tag{6.3-5}$$

The force is attractive in nature.

If the charge q is very close to the sphere, i.e., $d = a + \delta(\delta \ll a)$ then

$$\vec{F} \to -\frac{1}{4\pi\epsilon_0}\cdot\frac{q^2}{(2\delta)^2}\hat{k} \tag{6.3-6}$$

This result is similar to the case of a point charge placed in front of a conducting plane.

Induced charges

The surface density of induced charges at any point $C(a,\theta)$ on the sphere is given by

$$\begin{aligned} \sigma(\theta) &= -\epsilon_0\frac{\partial\phi}{\partial r}\bigg|_{r=a} \\ &= -\frac{q}{4\pi a}\cdot\frac{d^2-a^2}{(a^2+d^2-2ad\cos\theta)^{3/2}} \quad \text{[Using Eq. (6.3-3)]} \end{aligned} \tag{6.3-7}$$

As expected the induced charge density is negative. Note that $(a^2+d^2-2ad\cos\theta)^{3/2} = AC^3$. Thus, we can say that charge density at a point is inversely proportional to the cube of the distance of the point from q. The charge density is thus maximum at $\theta = 0°$ and minimum at $\theta = 180°$.

$$\sigma_{\max} = -\frac{q}{4\pi a}\cdot\frac{d+a}{(d-a)^2} \quad \text{and} \quad \sigma_{\min} = -\frac{q}{4\pi a}\frac{d-a}{(d+a)^2}. \tag{6.3-8}$$

Note that σ cannot be zero at any point on the sphere.

Total induced charge on the sphere can be obtained by integrating the charge density over the surface of the sphere. Thus, total induced charge

$$Q = \int_0^\pi \sigma(\theta)\cdot 2\pi a\sin\theta\cdot a d\theta = -\frac{q\,(d^2-a^2)\,a}{2}\int_0^\pi \frac{\sin\theta\, d\theta}{(a^2+d^2-2ad\cos\theta)^{3/2}}.$$

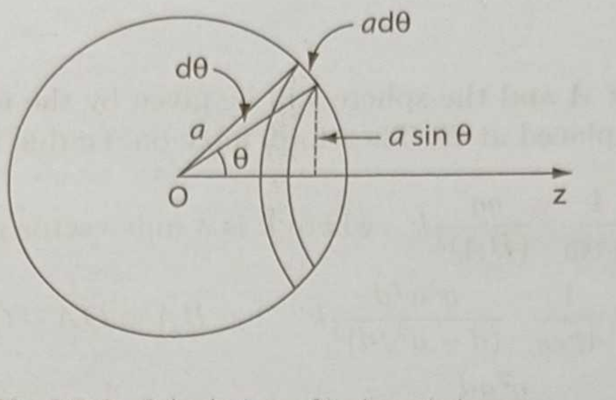


Fig 6.3-3: Calculation of induced charge.

Putting $a^2 + d^2 - 2ad\cos\theta = p^2$ we get

$$Q = -\frac{q\left(d^2 - a^2\right)}{2d}\int_{d-a}^{d+a}\frac{pdp}{p^3} = \frac{q\left(d^2 - a^2\right)}{2d}\left[\frac{1}{p}\right]_{d-a}^{d+a} = -\frac{qa}{d}, \tag{6.3-9}$$

which is, as expected, equal to the image charge.

6.4 A Point Charge and an Uncharged Insulated Conducting Sphere

Suppose a point charge q is placed at a distance d from the centre an insulated conducting sphere of radius a. The geometry of the problem is sketched in Fig 6.4-1. The potential ϕ at any external point $P(r,\theta)$ will be partly due to q and partly due to induced charges on the sphere. To solve the problem by the method of images we are to replace the induced charges by suitable image charges. In this attempt the following conditions must be satisfied:

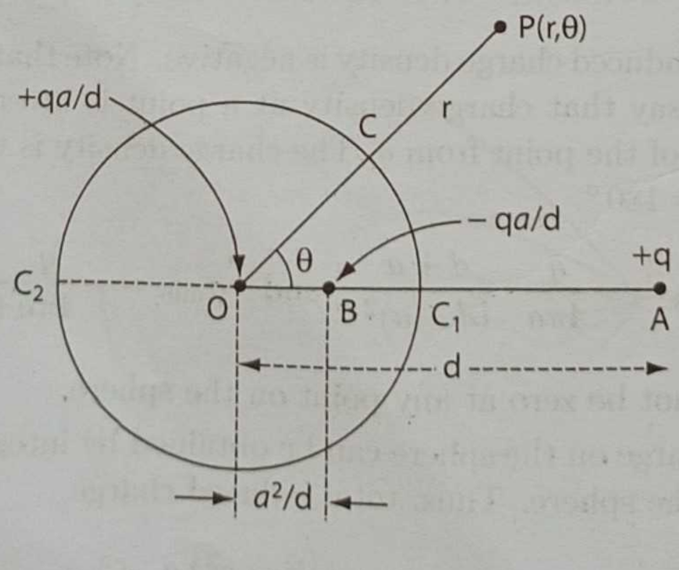


Fig 6.4-1